Homework Assignment # 6.

Plan for next week: Algebraic closures, splitting fields and normal extensions (online lectures 15,16,17 and § 13.4 in DF).

Problems, to be submitted by Thursday, March 7th.

Problem 1: Let F be an algebraically closed field of characteristic zero, V a vector space over F and $T : V \to V$ a linear map with $T^n = I$ for some n.

(a) Assume that $\dim(V) < \infty$. Use JCF to prove that T is diagonalizable

(b) Now do not assume that $\dim(V) < \infty$. Prove that T is still diagonalizable (by definition this means that V has a basis consisting of eigenvectors for T). **Hint:** All you need to use is that there is a polynomial $p(x) \in F[x]$ such that $p(T) = 0$ and p is separable, that is, p is a product of distinct linear factors.

Problem 2:

(a) Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Prove that $[K : \mathbb{Q}] = 4$.

(b) Let $L = \mathbb{Q}(\sqrt{2} + \sqrt{3})$. Prove that $L = K$ and hence $[L : \mathbb{Q}] = 4$ by (a).

Problem 3: Let $S = \{n_1, \ldots, n_k\}$ be a finite set of positive integers ≥ 2 and let $K = \mathbb{Q}(\sqrt{n_1}, \ldots, \sqrt{n_k})$.

(a) Prove that $[K : \mathbb{Q}] = 2^m$ for some $m \leq k$ and the set $P(S) = \{1\} \cup \{\sqrt{n} : n \text{ is a product of distinct elements of } S\}$ spans K over $\mathbb{Q}$.

(b) For each $0 \leq j \leq k$ let $\mathbb{Q}_j = \mathbb{Q}(\sqrt{n_1}, \ldots, \sqrt{n_j})$ (we set $\mathbb{Q}_0 = \mathbb{Q}$). Prove that $[K : \mathbb{Q}] < 2^k$ if and only if n_1 is a complete square or there exists $2 \leq i \leq k$ s.t. $\sqrt{n_i} = a + b\sqrt{n_{i-1}}$ for some $a, b \in \mathbb{Q}_{i-2}$.

(c) Assume that the elements of S are pairwise relatively prime and none of them is a complete square. Prove that $[K : \mathbb{Q}] = 2^k$. **Hint:** Use (b) and induction on $k = |S|$.

Problem 4: Let F be a field, and let α be an algebraic element of odd degree over F (recall that the degree of α over F is equal to $[F(\alpha) : F]$). Prove that $F(\alpha^2) = F(\alpha)$.

Problem 5: Let K/F be a finite field extension, $n = [K : F]$, and fix some basis $\Omega = \{\alpha_1, \ldots, \alpha_n\}$ for K over F. For any $\alpha \in K$ define $T_\alpha : K \to K$ by $T_\alpha(\beta) = \alpha\beta$. Note that $T_\alpha \in End_F(K)$. Let $A_\alpha = [T_\alpha]_\Omega \in Mat_n(F)$ be the matrix of T_α with respect to Ω.

(a) (practice) Prove that the map $K \to Mat_n(F)$ given by $\alpha \mapsto A_\alpha$ is an injective ring homomorphism.

(b) Prove that the minimal polynomial of α over F and the minimal polynomial of A_α coincide.

(c) Find the minimal polynomial of the matrix

$$A = \begin{pmatrix} 0 & 3 & 5 & 0 \\ 1 & 0 & 0 & 5 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

without doing extensive computations. **Hint:** Find an extension $K/\mathbb{Q}$ of degree 4, a basis for K over $\mathbb{Q}$ and an element $\alpha \in K$ such that $A_\alpha = A$ in the notations of part (a).

Problem 6: Before doing this problem read about composite fields on pp. 528-529. Let E/F be a field extension, let K_1 and K_2 be subfields of E containing F, and assume that the extensions K_1/F and K_2/F are finite. Let K_1K_2 be the composite field of K_1 and K_2. Prove that the F-algebra $K_1 \otimes_F K_2$ is a field if and only if $[K_1K_2 : F] = [K_1 : F][K_2 : F]$. **Hint:** First show that there exists an F-linear map $\Phi : K_1 \otimes_F K_2 \to K_1K_2$ such that $\Phi(a \otimes b) = ab$ for any $a \in K_1$ and $b \in K_2$. Then explain why Φ is always surjective.