**Homework Assignment # 4.**

**Plan for next week:** Classification of finitely generated modules over PIDs (12.1, online lecture 9). Rational Canonical Form (12.2, online lectures 10-11).

**Problems, to be submitted by Thu, February 14th.**

**Note on hints:** All hints are given at the end of the assignment, each on a separate page. Problems (or parts of problems) for which hint is available are marked with *.

**Problem 1*.** Prove Theorem N1 from Lecture 8: Let $R$ be a ring (with 1), let $M$ be an $R$-module and $N$ its submodule. Prove that $M$ is Noetherian $\iff$ $N$ and $M/N$ are both Noetherian.

**Problem 2.** Let $A$ be a ring (with 1). A subring $B$ of $A$ is called a *retract* if there exists a surjective ring homomorphism $\varphi : A \to B$ such that $\varphi_{|B} = id_B$, that is, $\varphi(b) = b$ for all $b \in B$.
Now let $M$ and $N$ be two $R$-modules. Prove that the tensor algebra $T(M)$ is (naturally isomorphic to) a subalgebra of $T(M \oplus N)$ and that this subalgebra is a retract. Also prove the analogous statement about the symmetric algebras.

**Problem 3:** Define the rank of an $R$-module $M$, denoted by $\mathrm{rk}(M)$, to be the minimal number of generators (WARNING: this definition is different from the definition in DF; the two definitions coincide for free modules over commutative rings).

(a)* Let $R$ be a PID, $M$ be a finitely generated $R$-module and $R/a_1R \oplus \ldots \oplus R/a_mR \oplus R^s$ its invariant factor decomposition, that is, $a_1, \ldots, a_m$ are nonzero non-units and $a_1 \mid a_2 \mid \ldots \mid a_m$. Prove that $\mathrm{rk}(M) = m + s$. **Warning:** It is not true in general that $\mathrm{rk}(P \oplus Q) = \mathrm{rk}(P) + \mathrm{rk}(Q)$.

(b) Again let $R$ be a PID. Let $F$ be a free $R$-module of rank $n$ with basis $e_1, \ldots, e_n$, let $N$ be the submodule of $F$ generated by some elements $v_1, \ldots, v_n \in F$, and let $A \in Mat_n(F)$ be the matrix such that

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = A \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$$

Find a simple condition on the entries of $A$ which holds if and only if $\mathrm{rk}(F/N) = n$.

**Problem 4**. Let $R = \mathbb{R}[x]$, $F = R^3$ (the standard 3-dimensional $R$-module) and $N$ the $R$-submodule of $F$ generated by $(1 - x, 1, 0)$, $(-2, 4 - x, 0)$ and $(1, -5, -x)$.

(a) Find compatible bases for $F$ and $N$, that is, bases satisfying the conclusion of the submodule structure theorem. **Note:** an algorithm for computing such bases is given in Lecture 8.

(b) Describe the quotient module $F/N$ in IF and ED forms.

**Problem 5:** Let $R$ be a commutative ring (with 1).

(a) Let $C$ be an $R$-algebra and let $A$ and $B$ be $R$-subalgebras of $C$ which commute with each other, that is, $ab = ba$ for any $a \in A, b \in B$ (note that $A$ and $B$ themselves do not have to be commutative). Prove that there is an $R$-algebra homomorphism $\varphi : A \otimes_R B \to C$ such that $\varphi(a \otimes b) = ab$ for each $a \in A$ and $b \in B$.

(b)* Now assume that $R$ is a field, and let $A$ be a finite-dimensional $R$-algebra. Prove that the algebra $A \otimes_R A$ cannot be a field unless $\dim_R A = 1$.

**Hint for 1:** The forward direction is easy. For the backwards direction, observe that if $\{P_i\}$ is an ascending chain of submodules of $M$, then $\{P_i \cap N\}$ is an ascending chain of submodules of $N$ and $\{(P_i + N)/N\}$ is an ascending chain of submodules of $M/N$.

**Hint for 3(a):** Let $p$ be a prime dividing $a_1$. How is $M$ related to $M' = (R/pR)^{m+s}$ and what is $\operatorname{rk}(M')$ (and why)?

**Hint for 5(b):** use 5(a).