Homework Assignment # 2.

Plan for next week: Tensor products of algebras, continued (end of 10.4, online lecture 5), Tensor, symmetric an exterior algebras (11.5, online lecture 6), modules over PID (12.1, online lecture 7).

Problems, to be submitted by Thu, January 31st.

Problem 0. Read the section on graded algebras from online Lecture 5 before the class on Tuesday, Jan 29.

Problem 1. Let R be a commutative ring. An R-module M is called *torsion* if for any $m \in M$ there exists nonzero $r \in R$ such that $rm = 0$. An R-module M is called *divisible* if for any nonzero $r \in R$ we have $rM = M$. In other words, M is divisible if for any $m \in M$ and nonzero $r \in R$ there exists $x \in M$ such that $rx = m$.
(a) Suppose that M is a torsion R-module and N is a divisible R-module. Prove that $M \otimes_R N = \{0\}$.
(b) Let $M = \mathbb{Q}/\mathbb{Z}$ considered as a $\mathbb{Z}$-module. Prove that $M \otimes_{\mathbb{Z}} M = \{0\}$.

Problem 2. Let R be a commutative ring, $\{N_\alpha\}$ a collection of R-modules and M another R-module.
(a) (practice, [DF, problem 14, p.376]) Prove that $M \otimes (\oplus N_\alpha) \cong \oplus (M \otimes N_\alpha)$ as R-modules (tensor products are over R).
(b) (see [DF, problem 15, p. 376]) Show by example that $M \otimes (\prod N_\alpha)$ need not be isomorphic to $\prod (M \otimes N_\alpha)$. **Hint:** Use the result of one of the previous problems on p. 376.

Problem 3. Let R be a commutative domain, and let M be a free R-module with basis $e_1, \ldots, e_k$. Prove that the element $e_1 \otimes e_2 + e_2 \otimes e_1 \in M \otimes M$ is not representable as a simple tensor $m \otimes n$ for some $m, n \in M$.

Problem 4. Problem 16 on p.376 of DF

Problem 5. Problem 17 on pp.376-377 of DF.

Problem 6. (practice) Problem 21 on p.377 of DF

Problem 7. Let R be a commutative ring with 1, let S and B be R-algebras, and suppose that S is commutative. Then $S \otimes_R B$ is an R-algebra (and in particular a ring) and also an S-module via the extension of scalars construction (type I tensor product). Prove that $S \otimes_R B$ is also an S-algebra.

Problem 8. Problem 25 on pp. 377 of DF (the S-algebra structure on $S \otimes_R R[x]$ is described in Problem 7).

Problem 9. (a) Let V be a finite-dimensional vector space over $\mathbb{C}$ (complex numbers). Note that V can also be considered as a vector space over $\mathbb{R}$, but $\dim_{\mathbb{R}}(V) = 2\dim_{\mathbb{C}}(V)$. Prove that $V \otimes_{\mathbb{C}} V$ is not isomorphic to $V \otimes_{\mathbb{R}} V$ as vector spaces over $\mathbb{R}$ and compute their dimensions over $\mathbb{R}$.
(b) Let R be an integral domain and F its field of fractions. Prove that $F \otimes_R F \cong F \otimes_F F \cong F$ as F-algebras (again the F-algebra structure on structure on $F \otimes_R F$ is described in Problem 7).