Homework Assignment # 11.

Plan for the next week: Group cohomology (Chapter 17)

Problems, to be submitted by Thu, April 25th.

1. This is a continuation of Problem 1 from Midterm#2. Let p be a prime, with $p \equiv 3 \mod 4$, $\omega = e^{2\pi i/p}$, $K = \mathbb{Q}(\omega)$ and L the unique subfield of K with $[L : \mathbb{Q}] = 2$. Let S be the set of elements of $(\mathbb{Z}/p\mathbb{Z})^{\times}$ representable as squares and T the set of elements of $(\mathbb{Z}/p\mathbb{Z})^{\times}$ not representable as squares.

(a) Prove that any $\alpha \in K$ can be uniquely represented as $\alpha = \sum_{s\in S} b_s\omega^s + \sum_{t\in T} c_t\omega^t$, with $b_s, c_t \in \mathbb{Q}$.

(b) Let $\alpha \in K$. Prove that $\alpha \in L$ if and only if in the above decomposition of α all b_s are the same and all c_t are the same.

(c) Let $\zeta = \sum_{s\in S} \omega^s$, $\eta = \zeta\overline{\zeta}$, and write $\eta = \sum_{s\in S} b_s\omega^s + \sum_{t\in T} c_t\omega^t$ as in (a). Prove that

  (i) there exists $d \in \mathbb{Q}$ such that $b_s = c_t = d$ for all s and t and

  (ii) $\sum_{s\in S} b_s + \sum_{t\in T} c_t = (p-1)^2/4 - p \cdot (p-1)/2 = -(p-1)(p+1)/4$

(d) Use (c) to prove that $\eta = (p + 1)/4$ and deduce that $L = \mathbb{Q}(\sqrt{-p})$.

Problem 2:

(a) DF, Problem 3 on page 403

(b) Let R be a PID. Prove that every finitely generated projective module over R is free (the assertion is true for infinitely generated projective modules as well, but the general case probably cannot be solved using just the results we have discussed)

Problem 3: Let F be a field, $n \geq 2$ an integer and $R = Mat_n(F)$. Let $M = F^n$, and consider M as a left R-module (with respect to left multiplication). Prove that M is projective, but not free.

Problem 4: Let $R = \mathbb{Z}[\sqrt{-5}]$ be a field and $I = (2, 1 + \sqrt{-5})$, and define $f : R^2 \to I$ by $f(x, y) = 2x + (1 + \sqrt{-5})y$ (note that f is surjective).

(a) Show that the exact sequence $0 \to \operatorname{Ker} f \to R^2 \to I \to 0$ splits by explicitly constructing a splitting. Deduce that I is a projective R-module. **Hint:** there is a splitting of the form $g(x) = (ax, bx)$ for suitable $a, b \in \operatorname{Frac}(R) = \mathbb{Q}[\sqrt{-5}]$.

(b) Prove that I is not a principal ideal and deduce that I is not a free R-module.

Problem 5: DF, Problem 4 on page 403

Problem 6: Let R be a commutative ring with 1. An R-module M is called *torsion-free* if for any $r \in R$ and $m \in M$, the equality $rm = 0$ implies that $m = 0$ or $r = 0$ or r is a zero divisor. Prove that any flat R-module is torsion-free. **Hint:** argue by contrapositive.