Homework Assignment # 10.

Plan for the next week: Section 10.5 (exact sequences and injective and projective modules)

Problems, to be submitted by Thu, April 18th.

Problem 1: Let $f(x)$ and $g(x)$ be irreducible polynomials in $\mathbb{F}_p[x]$ of the same degree and let $F = \mathbb{F}_p[x]/(f(x))$. Prove that $g(x)$ splits completely over F.

Problem 2: Let p be a prime, n a positive integer and $\Phi_n(x) = x^{p^n} - x \in \mathbb{F}_p[x]$. Prove that $\Phi_n(x)$ is equal to the product of all monic irreducible polynomials in $\mathbb{F}_p[x]$ whose degrees divide n (where each polynomial occurs with multiplicity one).

Problem 3: Prove the following analogue of Kummer's theorem for abelian extensions: Let $n \in \mathbb{N}$ and let F be a field containing primitive n^{th} root of unity.

(a) Let K/F be a finite Galois extension such that $\mathrm{Gal}\,(K/F)$ is abelian of exponent n. Then there exists $a_1, \dots, a_t \in K$ s.t. $K = F(\sqrt[n]{a_1}, \dots, \sqrt[n]{a_t})$, or more precisely, there exists $\alpha_1, \dots, \alpha_t \in K$ s.t. $K = F(\alpha_1, \dots, \alpha_t)$ and $\alpha_i^n \in F$ for all i.

(b) Conversely, suppose that $K = F(\sqrt[n]{a_1}, \dots, \sqrt[n]{a_t})$ for some $a_1, \dots, a_t \in F$. Prove that K/F is Galois, and $\mathrm{Gal}\,(K/F)$ is abelian of exponent n.

Problem 4: (practice). DF, problem 18 on p.583. **Note:** Make sure to solve this problem if you did not solve Problem 17 completely correctly.

Problem 5: DF, problem 21 on p.583.

Problem 6: Let p be a prime, $K = \mathbb{F}_p(t)$, the field of rational functions in one variable over $\mathbb{F}_p$. Let $\sigma : K \to K$ be the unique automorphism of K such that $\sigma(t) = t + 1$ and $G = \langle \sigma \rangle$ (clearly G is cyclic of order p). Let $F = K^G$. Find an explicit element s such that $F = \mathbb{F}_p(s)$ and prove your answer.

Problem 7: DF, problem 9 on p.636.