Homework Assignment # 7.

Plan for next week: start Galois theory (online lecture 19, DF 14.1 and parts of 14.2)

Problems, to be submitted by Thursday, March 22nd.

Problem 1: Let F be a field and Ω a subset of $F[x]$. Use the existence and uniqueness of algebraic closures to prove that there exists a unique splitting field for Ω over F up to F-isomorphism. **Hint:** First show that any splitting field K for Ω lies in some algebraic closure of F.

Problem 2: Before doing this problem read § 13.6 in DF. Let p be a prime, $n \geq 2$ an integer, $f(x) = x^n - p$, and let $K \subset \mathbb{C}$ be the splitting field for $f(x)$ over $\mathbb{Q}$. As we proved in class, $K = \mathbb{Q}(\sqrt[n]{p}, \omega_n)$ where $\omega_n = e^{2\pi i/n}$.

(a) Prove that $[K : \mathbb{Q}] \leq n\varphi(n)$ where φ is the Euler function.

(b) Assume that n is prime. Prove that inequality in (a) is equality.

(c) Let $p = 3$ and $n = 12$. Prove that inequality in (a) is strict and find $[K : \mathbb{Q}]$. **Hint:** Compute ω_{12} explicitly.

Problem 3 (practice): (a) Prove that if $K/\mathbb{Q}$ is any field extension, then any automorphism of K must fix $\mathbb{Q}$ elementwise.
(b) Prove that the automorphism group $\mathrm{Aut}(\mathbb{R}/\mathbb{Q})$ is trivial. See [DF, Problem 7, p.567] for a sketch of the proof.

Problem 4: Let K/F be an algebraic extension. Prove that K/F is normal if and only if for any algebraic extension L/K and any F-automorphism $\sigma \in \mathrm{Aut}_F(L)$ we have $\sigma(K) = K$.

Problem 5: Let K/F be a field extension, and let K_1 and K_2 be subfields of K containing F such that the extensions K_1/F and K_2/F are normal. Prove that the extensions K_1K_2/F and $K_1 \cap K_2/F$ are also normal.

Problem 6: Problems 10 and 11 in [DF, p.556]

Problem 7: Let $p_1, \ldots, p_k$ be distinct primes and $K = \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_k})$. Find a primitive element of $K/\mathbb{Q}$ (and justify that your element is indeed primitive).

Problem 8 (practice): Problem 13 in [DF, p.556].