Homework Assignment # 6.

Plan for next week: Normal and separable extensions (online lectures 17,18, § 13.5 and parts of § 13.4 in DF)

Problems, to be submitted by Thursday, March 15th.

Problem 1: V be a vector space over an algebraically closed field F, let $n = \dim V$ and $T \in \mathfrak{gl}(V)$. Recall that if λ is an eigenvalue of T, a sequence of vectors $v_1, \ldots, v_k \in V$ is called a *cycle of generalized eigenvectors corresponding to* λ if $(T - \lambda)v_1 = 0$ and $(T - \lambda)v_i = v_{i-1}$ for $i > 1$. The vector v_1 is called the *initial vector* of the cycle and v_n is called the *tail vector*.
Note that a basis Ω of V is a Jordan basis for $T \iff \Omega$ is an ordered union of cycles of generalized eigenvectors (with one cycle corresponding to each Jordan block).

(a) Let $v_1, \ldots, v_k \in V$ be a cycle of generalized eigenvectors with $v_1 \neq 0$. Prove that $v_1, \ldots, v_k$ are linearly independent.

(b) Suppose that T has unique eigenvalue λ and just one Jordan block, and let $S = T - \lambda I$. Let v be any vector in $V \setminus \mathrm{Ker}\,(S^{n-1})$. Deduce that $\{S^{n-1}v, \ldots, Sv, v\}$ is a Jordan basis for T.

(c) (practice) Let $C_1, \ldots, C_m$ be cycles of generalized eigenvectors (possibly corresponding to different eigenvalues), and suppose that the initial vectors of these cycles are linearly independent. Prove that the cycles $C_1, \ldots, C_m$ are disjoint and their union is linearly independent. **Hint:** Since $V = \oplus_{\lambda \in Spec(T)} V_\lambda$ where V_λ is the root subspace of T corresponding to λ and any generalized cycle of eigenvectors corresponding to λ is obviously contained in V_λ, without loss of generality you may assume that all cycles $C_1, \ldots, C_m$ correspond to the same eigenvalue.

Problem 2: Again let V be a finite-dimensional vector space over an algebraically closed field, $T \in \mathfrak{gl}(V)$ and $n = \dim V$.

(a) Assume that T has unique eigenvalue 0 and two Jordan blocks: a 1×1 block and a 2×2 block (so $n = 3$). Use Problem 1 to justify the following algorithm for computing a Jordan basis for T: Take any $v \in V \setminus \mathrm{Ker}\,(T)$ and choose $w \in \mathrm{Ker}\,(T)$ such that $\{w, Tv\}$ is a basis for $\mathrm{Ker}\,(T)$ (why is this possible?); then $\{w, Tv, v\}$ is a Jordan basis for T.

(b) Assume that T has unique eigenvalue 0 and two Jordan blocks, both of which are 2×2 (so $n = 4$). State and justify an algorithm for finding a Jordan basis similar to the one in (a).

(c) Assume that for each $\lambda \in Spec(T)$ there is only one Jordan λ-block in $JCF(T)$. Decribe an algoirhtm for computing a Jordan basis of T. **Hint:** You just need a minor variation of the algorithm in Problem 1(b).

Remark: Recall that if K/F and L/K are finite field extensions, then $[L : F] = [L : K][K : F]$. In particular, both $[K : F]$ and $[L : K]$ divide $[L : F]$. This simple observation turns out to be extremely useful.

Problem 3: (a) Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Prove that $[K : \mathbb{Q}] = 4$.

(b) Let $L = \mathbb{Q}(\sqrt{2} + \sqrt{3})$. Prove that $L = K$ and hence $[L : \mathbb{Q}] = 4$ by (a).

Problem 4: Let $S = \{n_1, \ldots, n_k\}$ be a finite set of positive integers ≥ 2 and let $K = \mathbb{Q}(\sqrt{n_1}, \ldots, \sqrt{n_k})$.

(a) Prove that $[K : \mathbb{Q}] = 2^m$ for some $m \leq k$ and the set $P(S) = \{1\} \cup \{\sqrt{n} : n \text{ is a product of distinct elements of } S\}$ spans K over $\mathbb{Q}$.

(b) For each $0 \leq j \leq k$ let $\mathbb{Q}_j = \mathbb{Q}(\sqrt{n_1}, \ldots, \sqrt{n_j})$ (we set $\mathbb{Q}_0 = \mathbb{Q}$). Prove that $[K : \mathbb{Q}] < 2^k$ if and only if n_1 is a complete square or there exists $2 \leq i \leq k$ s.t. $\sqrt{n_i} = a + b\sqrt{n_{i-1}}$ for some $a, b \in \mathbb{Q}_{i-2}$.

(c) Assume that the elements of S are pairwise relatively prime and none of them is a complete square. Prove that $[K : \mathbb{Q}] = 2^k$. **Hint:** Use (b) and induction on $k = |S|$.

Problem 5: Let F be a field, and let α be an algebraic element of odd degree over F (where the degree of α over F is $[F(\alpha) : F]$). Prove that $F(\alpha^2) = F(\alpha)$.

Problem 6: Let K/F be a finite field extension, $n = [K : F]$, and fix some basis $\Omega = \{\alpha_1, \ldots, \alpha_n\}$ for K over F. For any $\alpha \in K$ define $T_\alpha : K \to K$ by $T_\alpha(\beta) = \alpha\beta$. Note that $T_\alpha \in End_F(K)$. Let $A_\alpha = [T_\alpha]_\Omega \in Mat_n(F)$ be the matrix of T_α with respect to Ω.

(a) (practice) Prove that the map $K \to Mat_n(F)$ given by $\alpha \mapsto A_\alpha$ is an injective ring homomorphism.

(b) Prove that the minimal polynomial of α over F and the minimal polynomial of A_α coincide.

(c) Find the minimal polynomial of the matrix

$$A = \begin{pmatrix} 0 & 3 & 5 & 0 \\ 1 & 0 & 0 & 5 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

without doing extensive computations. **Hint:** Find an extension $K/\mathbb{Q}$ of degree 4, a basis for K over $\mathbb{Q}$ and an element $\alpha \in K$ such that $A_\alpha = A$ in the notations of part (a).

Problem 7: Before doing this problem read about composite fields on pp. 528-529. Let E/F be a field extension, let K_1 and K_2 be subfields of E containing F, and assume that the extensions K_1/F and K_2/F are finite. Let K_1K_2 be the composite field of K_1 and K_2. Prove that the F-algebra $K_1 \otimes_F K_2$ is a field if and only if $[K_1K_2 : F] = [K_1 : F][K_2 : F]$. **Hint:** First show that there exists an F-linear map $\Phi : K_1 \otimes_F K_2 \to K_1K_2$ such that $\Phi(a \otimes b) = ab$ for any $a \in K_1$ and $b \in K_2$. Then explain why Φ is always surjective.