Homework Assignment # 5.

Plan for next week: Jordan canonical form (12.3 and online lecture 12+) and field extensions (13.1, 13.2 and online lecture 14).

Problems, to be submitted by Thu, February 23rd.

Problem 1: Let R be a commutative ring (with 1).

(a) Let C be an R-algebra and let A and B be R-subalgebras of C which commute with each other, that is, $ab = ba$ for any $a \in A, b \in B$ (note that A and B themselves do not have to be commutative). Prove that there is an R-algebra homomorphism $\varphi : A \otimes_R B \to C$ such that $\varphi(a \otimes b) = ab$ for each $a \in A$ and $b \in B$.

(b) Prove that $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{C}$ as rings (as usual $\mathbb{R}$ is real numbers and $\mathbb{C}$ are complex numbers).

(c) Now assume that R is a field, and let A be a finite-dimensional R-algebra. Prove that the algebra $A \otimes_R A$ cannot be a field unless $\dim_R A = 1$. **Hint:** use (a).

Problem 2: DF, Problem 6, page 488.

Problem 3: DF, Problem 9, page 489, first two matrices are for practice.

Problem 4: Recall that for a matrix A we denoted by $\chi_A(x)$ and $\mu_A(x)$ its characteristic and minimal polynomials, respectively. Determine the number of possible RCFs of 8×8 matrices A over $\mathbb{Q}$ with $\chi_A(x) = x^8 - x^4$. Explain your argument in detail.

Problem 5: (a) (practice) Prove that two 3×3 matrices over some field F are similar if and only if they have the same minimal and characteristic polynomials. Give an example showing that this does not hold for 4×4 matrices.

(b) A matrix A is called idempotent if $A^2 = A$. Prove that two idempotent $n \times n$ matrices are similar if and only if they have they same rank. **Hint:** What is the minimal polynomial of an idempotent matrix? How does rank relate to eigenvalue 0?

Problem 6: Find the number of distinct conjugacy classes in the group $GL_3(\mathbb{Z}/2\mathbb{Z})$ and specify one element in each conjugacy class.