Homework Assignment # 4.

Plan for next week: Rational Canonical Form (12.2, online lectures 10-11).

Problems, to be submitted by Thu, February 16th.

Problem 1. (a) Let R be a commutative ring, let M be an R-module and N its submodule. Prove that M is Noetherian $\iff$ N and M/N are both Noetherian.

Hint: The forward direction is easy. For the backwards direction, observe that if $\{P_i\}$ is an ascending chain of submodules of M, then $\{P_i \cap N\}$ is an ascending chain of submodules of N and $\{(P_i + N)/N\}$ is an ascending chain of submodules of M/N.

(b) Let R be a commutative Noetherian ring. Use (a) to prove that R^n is a Noetherian module for any $n \in \mathbb{N}$.

(c) Use (a) and (b) to prove Lemma from class: if R is Noetherian, then every submodule of a finitely generated R-module is finitely generated.

Problem 2. Let A be a ring (with 1). A subring B of A is called a *retract* if there exists a surjective ring homomorphism $\varphi : A \to B$ such that $\varphi_{|B} = id_B$, that is, $\varphi(b) = b$ for all $b \in B$.

Now let M and N be two R-modules. Prove that the tensor algebra $T(M)$ is (naturally isomorphic to) a subalgebra of $T(M \oplus N)$ and that this subalgebra is a retract. Also prove the analogous statement about the symmetric algebras.

Problem 3. (This is the first half of the practice problem 2.7) Let I and J be ideals of a (commutative) ring R, and let $\pi_I : R \to R/I$ and $\pi_J : R \to R/J$ be canonical projections.

(a) Prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor $\pi_I(1) \otimes \pi_J(r)$ for some $r \in R$ and also as $\pi_I(r') \otimes \pi_J(1)$ for some $r' \in R$.

(b) Use (a) to prove that $R/I \otimes_R R/J \cong R/(I + J)$ (as R-modules).

Problem 4: Let R be a PID. For an R-module M define $\mathrm{rk}(M)$ to be the minimal size of a generating set of M.

(a) Let M be a finitely generated R-module and $R/a_1R \oplus \ldots \oplus R/a_mR \oplus R^s$ its invariant factor decomposition, that is, $a_1, \ldots, a_m$ are nonzero non-units and $a_1 \mid a_2 \mid \ldots \mid a_m$. Prove that $\mathrm{rk}(M) = m + s$. **Warning:** It is not true in general that $\mathrm{rk}(P \oplus Q) = \mathrm{rk}(P) + \mathrm{rk}(Q)$. **Hint:** Let p be a prime dividing a_1. How is M related to $M' = (R/pR)^{m+s}$ and what is $\mathrm{rk}(M')$ (and why)?

(b) Let F be a free R-module of rank n with basis $e_1, \ldots, e_n$, let N be the submodule of F generated by some elements $v_1, \ldots, v_n \in F$, and let $A \in Mat_n(F)$ be the matrix such that

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = A \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$$

Find a simple condition on the entries of A which holds if and only if $\mathrm{rk}(F/N) = n$.

Problem 5. Let $R = \mathbb{R}[x]$, $F = R^3$ (the standard 3-dimensional R-module) and N the R-submodule of F generated by $(1 - x, 1, 0)$, $(-2, 4 - x, 0)$ and $(1, -5, -x)$.

(a) Find compatible bases for F and N, that is, bases satisfying the conclusion of the submodule structure theorem.

(b) Describe the quotient module F/N in IF and ED forms.