Homework Assignment # 3.

Plan for next week: Modules over PID continued (12.1, online lectures 7-9).

Problems, to be submitted by Thu, February 9th.

Problem 1. The main goal of this problem is to classify 2-dimensional $\mathbb{R}$-algebras ($\mathbb{R}$=reals), that is, $\mathbb{R}$-algebras which are 2-dimensional as vector spaces over $\mathbb{R}$. Note that the idea of solution was discussed in the last lecture of Algebra-I.

Let F be a field with $\mathrm{char}(F) \neq 2$, and let A be a 2-dimensional F-algebra (as always, with 1).

(a) Let $u \in A$ be any element which is not an F-multiple of 1. Prove that

  (i) u generates A as an F-algebra, that is, the minimal F-subalgebra of A containing u and 1 is A itself.

  (ii) u satisfies a quadratic equation $au^2+bu+c=0$ for some $a, b, c \in F$ with $a \neq 0$.

(b) Show that there exists $v \in A$ such that $v^2 \in F$. **Hint:** take any u as in (a), and look for v of the form $u + \beta$ with $\beta \in F$.

(c) Deduce from (b) that A is isomorphic as an F-algebra to $F[x]/(x^2 - c)$ for some $c \in F$.

(d) Prove that if $c = d^2$ for some $d \in F$, then $F[x]/(x^2 - c) \cong F \times F$.

(e) Now let $F = \mathbb{R}$ (real numbers). Prove that in (c) one can choose $c = 0, 1$ or -1. Then prove that the algebras $\mathbb{R}[x]/(x^2+1)$, $\mathbb{R}[x]/(x^2-1)$ and $\mathbb{R}[x]/x^2$ are pairwise non-isomorphic. **Hint:** the algebras can be distinguished from each other by simple abstract properties.

Problem 2. Let R be a commutative ring with 1. A left R-module M is called *Noetherian* if it satisfies the ascending chain condition on submodules and *Artinian* if it satisfies the descending chain condition on submodules. Assume that an R-module M is both Artinian and Noetherian. (For example, R might be a field, and M might be a finite-dimensional vector space over R). Let $T : M \to M$ be an R-module homomorphism.

(a) Prove that there exists $k \in \mathbb{N}$ s.t. $\mathrm{Ker}\,(T^k) = \mathrm{Ker}\,(T^{2k})$ and $\mathrm{Im}(T^k) = \mathrm{Im}(T^{2k})$.

(b) Prove that if k is as in part (a), then $M = \mathrm{Ker}\,(T^k) \oplus \mathrm{Im}(T^k)$

(c) Deduce from (a) and (b) that there exist submodules M_0 and M_1 of M s.t. $M = M_0 \oplus M_1$, $T_{|M_0}$ is nilpotent and $T_{|M_1}$ is invertible (as a map from M_1 to M_1).

Problem 3. Let V and W be finite dimensional vector spaces over a field F, let $\{v_1, \ldots, v_n\}$ be a basis of V and $\{w_1, \ldots, w_m\}$ a basis of W.
Let $\varphi : V \otimes_F W \to Mat_{n\times m}(F)$ be the F-linear transformation such that $\varphi(v_i \otimes w_j) = e_{ij}$ where e_{ij} is the matrix whose (i,j)-entry is equal to 1 and all other entries are equal to 0 (note that such transformation exists and is unique because $\{v_i \otimes w_j : 1 \le i \le n, 1 \le j \le m\}$ is a basis for $V \otimes_F W$; furthermore, φ is an isomorphism since matrices $\{e_{ij}\}$ form a basis of $Mat_{n\times m}(F)$).
Prove that for a matrix $A \in Mat_{n\times m}(F)$ the following are equivalent:

(a) $A = \varphi(v \otimes w)$ for some $v \in V, w \in W$ (note: v and w need not be elements of the above bases)

(b) $rk(A) \le 1$.

Note that this problem yields a one-line solution to Problem 5 from HW#2.

Problem 4 (practice). Let $R = \oplus_{n=0}^{\infty} R_n$ be a graded ring. An element $r \in R$ is called *homogeneous* if $r \in R_n$ for some n.

Any $r \in R$ can be uniquely written as $r = \sum_{n=0}^{\infty} r_n$ where $r_n \in R_n$ and only finitely many r_n's are nonzero. The elements $\{r_n\}$ are called the *homogeneous components of* r.
(a) Let I be an ideal of R. Prove that the following are equivalent:

(i) I is a graded ideal, that is, $I = \oplus_{n=0}^{\infty} I \cap R_n$

(ii) For each $r \in I$ all homogeneous components of r also lie in I

(b) Let I be an ideal of R generated by homogeneous elements (possibly of different degrees). Prove that I is graded.

Problem 5.

(a) Let R be a commutative ring with 1 and M an $R-module$. Let $m_1, \ldots, m_k$ be elements of M and $\sigma \in S_k$ a permutation. Prove that $m_{\sigma(1)} \wedge \ldots \wedge m_{\sigma(k)} = (-1)^{\sigma} m_1 \ldots m_k$.

(b) Use (a) to prove Proposition 6.5 from the online version of Lecture 6.