Homework Assignment # 2.

Plan for next week: Tensor, symmetric an exterior algebras (11.5, online lecture 6), modules over PID (12.1, online lecture 7).

Problems, to be submitted by Thu, February 2nd.

Problem 0. Read the section on graded algebras from online Lecture 5 before the class on Tuesday, Jan 31.

Problem 1. Let R be a commutative ring. An R-module M is called *torsion* if for any $m \in M$ there exists nonzero $r \in R$ such that $rm = 0$. An R-module M is called *divisible* if for any nonzero $r \in R$ we have $rM = M$. In other words, M is divisible if for any $m \in M$ and nonzero $r \in R$ there exists $x \in M$ such that $rx = m$.
(a) Suppose that M is a torsion R-module and N is a divisible R-module. Prove that $M \otimes_R N = \{0\}$.
(b) Let $M = \mathbb{Q}/\mathbb{Z}$ considered as a $\mathbb{Z}$-module. Prove that $M \otimes_{\mathbb{Z}} M = \{0\}$.

Problem 2. Let R be a commutative ring, $\{N_\alpha\}$ a collection of R-modules and M another R-module.
(a) (practice, [DF, problem 14, p.376]) Prove that $M \otimes (\oplus N_\alpha) \cong \oplus(M \otimes N_\alpha)$ as R-modules (tensor products are over R).
(b) (see [DF, problem 15, p. 376]) Show by example that $M \otimes (\prod N_\alpha)$ need not be isomorphic to $\prod(M \otimes N_\alpha)$. **Hint:** Use the result of one of the previous problems on p. 376.

Problem 3. Let $R \subseteq S$ be rings (not necessarily commutative), and let $R[x]$ (resp. $S[x]$) be the ring of polynomials over R (resp. S) (we assume that x commutes with everything). Prove that

$$S \otimes_R R[x] \cong S[x] \text{ as } S\text{-modules}$$

Problem 4. (a) Let V be a finite-dimensional vector space over $\mathbb{C}$ (complex numbers). Note that V can also be considered as a vector space over $\mathbb{R}$, but $\dim_{\mathbb{R}}(V) = 2\dim_{\mathbb{C}}(V)$. Prove that $V \otimes_{\mathbb{C}} V$ is not isomorphic to $V \otimes_{\mathbb{R}} V$ as vector spaces over $\mathbb{R}$ and compute their dimensions over $\mathbb{R}$.
(b) Let R be an integral domain and F its field of fractions. Prove that $F \otimes_R F \cong F \otimes_F F \cong F$ as F-modules. Note that the F-module structure on $F \otimes_R F$ is given by the extension of scalars construction (type I tensor product).

Problem 5. Let R be a commutative domain, and let M be a free R-module with basis $e_1, \ldots, e_k$. Prove that the element $e_1 \otimes e_2 + e_2 \otimes e_1 \in M \otimes M$ is not representable as a simple tensor $m \otimes n$ for some $m, n \in M$.

Problem 6. Problem 17 on pp.376-377 of DF.

Problem 7 (practice). Let I and J be ideals of a (commutative) ring R, and let $\pi_I : R \to R/I$ and $\pi_J : R \to R/J$ be canonical projections.
(a) Prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor $\pi_I(1) \otimes \pi_J(r)$ for some $r \in R$.
(b) Prove that $R/I \otimes_R R/J \cong R/(I + J)$ (as R-modules).
(c) Show that there is a surjective R-module homomorphism $I \otimes_R J \to IJ$ such that $i \otimes j \mapsto ij$.
(d) Give an example where φ in (c) is not an isomorphism.

Problem 8. Let R be a commutative ring (with 1) and $n, m \in \mathbb{N}$. Prove that $R^n \otimes R^m \cong R^{nm}$ as R-algebras. As usual $R^k = \underbrace{R \oplus \ldots \oplus R}_{k \text{ times}}$.