Homework Assignment # 10.

Plan for the next week: Tuesday (April 17) – proof of Hilbert's Nullstellensatz; Thursday (April 19) – dimension theory of affine varieites. Good references on commutative algebra and algebraic geometry freely available online are notes by J. Milne

http://www.jmilne.org/math/xnotes/CA.pdf

and

http://www.jmilne.org/math/CourseNotes/AG.pdf

Problems, to be submitted by Thu, April 19th.

Problem 1: Let k be a field and $R = k[x_1, \ldots, x_n]$ for some $n \geq 1$.

(a) Prove that if $\{I_\alpha\}$ is any collection of ideals of R, then $\cap Z(I_\alpha) = Z(\cup I_\alpha) = Z(\sum I_\alpha)$.

(b) Prove that if I and J are ideals of R, then $Z(I) \cup Z(J) = Z(I \cap J) = Z(IJ)$.

Problem 2: Let k be an algebraically closed field. An algebraic set $V \subseteq k^n$ is called <u>irreducible</u> if $V \neq \emptyset$ and V cannot be written as the union $V = V_1 \cup V_2$ where V_1 and V_2 are both algebraic, with $V_1 \neq V$ and $V_2 \neq V$.

(a) Prove that V is irreducible if and only if its vanishing ideal $I(V)$ is prime.

(b) We will prove in class next week that any algebraic set V can be uniquely written as a union of finitely many algebraic subsets $V = \cup_{i=1}^{k} V_i$ where V_i's are irreducible and do not contain each other. Such V_i's are called <u>irreducible components of V</u>. Let

$$V = Z(xy - y, x^2 z - z) \subset k^3,$$

the set of common zeroes of $xy - y$ and $x^2 z - z$. Find irreducible components of V and their vanishing ideals. The answer will depend on $char(k)$.

Problem 3: DF, Problem 17 on pages 582-583 (this time really assigned).

Problem 4: This is a continuation of Problem 7 in HW#9.

(c) Suppose K/L and L/F are both p-extensions, and let M be the Galois closure of K over F (note: we do not know whether K/F is Galois or not). Prove that M/F is also a p-extension. **Hint:** first show that M/L is Galois.

(d) Now assume only that L/F is a separable extension with $[L : F]$ a power of p, and let M be the Galois closure of L over F. Prove that $[M : F]$ need not be a power of p.

Problem 5: Let $f(x)$ and $g(x)$ be irreducible polynomials in $\mathbb{F}_p[x]$ of the same degree and let $F = \mathbb{F}_p[x]/(f(x))$. Prove that $g(x)$ splits completely over F.

Problem 6: Let p be a prime, n a positive integer and $\Phi_n(x) = x^{p^n} - x \in \mathbb{F}_p[x]$. Prove that $\Phi_n(x)$ is equal to the product of all monic irreducible polynomials in $\mathbb{F}_p[x]$ whose degrees divide m (where each polynomial occurs with multiplicity one).

Problem 7: (practice) Prove the following analogue of Kummer's theorem for abelian extensions: Let $n \in \mathbb{N}$ and let F be a field containing primitive n^{th} root of unity.

(a) Let K/F be a finite Galois extension such that $\mathrm{Gal}\,(K/F)$ is abelian of exponent n. Then there exists $a_1, \ldots, a_t \in K$ s.t. $K = F(\sqrt[n]{a_1}, \ldots, \sqrt[n]{a_t})$, or more precisely, there exists $\alpha_1, \ldots, \alpha_t \in K$ s.t. $K = F(\alpha_1, \ldots, \alpha_t)$ and $\alpha_i^n \in F$ for all i.

(b) Conversely, suppose that $K = F(\sqrt[n]{a_1}, \ldots, \sqrt[n]{a_t})$ for some $a_1, \ldots, a_t \in F$. Prove that K/F is Galois, and $\mathrm{Gal}\,(K/F)$ is abelian of exponent n.