Homework Assignment # 1.

Plan for next week: Tensor products of modules and algebras. Reading: 10.4 in DF and Lectures 3, 4, 5 from the webpage. Note that the first part of Lecture 5 was discussed in the last class of Algebra-I.

Problems, to be submitted by Thu, January 26th.

Convention: All rings below are assumed to have 1, and all modules are left modules.

Problem 1: Let R be a commutative Noetherian ring and $\varphi : R \to R$ a surjective ring homomorphism. Prove that φ must be an isomorphism. **Hint:** Consider the ideals $\mathrm{Ker}\,(\varphi^n)$, $n \in \mathbb{N}$.

Problem 2 (practice): Let M be an R-module for some ring R (not necessarily commutative).

(a) For a subset N of M the *annihilator of N in R* is defined to be the set $Ann_R(N) := \{r \in R : rn = 0 \text{ for any } n \in N\}$. Prove that $Ann_R(N)$ is a left ideal of R.

(b) Prove that if N is a submodule of M, then $Ann_R(N)$ is an ideal of R (that is, a two-sided ideal).

(c) For a subset I of R the *annihilator of I in M* is defined to be the set $Ann_M(I) := \{m \in M : xm = 0 \text{ for any } x \in I\}$. Find a natural condition on I which guarantees that $Ann_M(I)$ is a submodule of M.

Problem 3: Let R be a ring and let M be an R-module.

(a) Prove that for any $m \in M$, the map $x \mapsto xm$ from R to M is a homomorphism of R-modules (recall that R is an R-module with the left multiplication action).

(b) Assume that R is commutative, and let M be an R-module. Prove that $Hom_R(R, M) \cong M$ as R-modules. **Hint:** An element of $Hom_R(R, M)$ is uniquely determined by where it maps 1.

Problem 4: In Lecture 27 we obtained a simple characterization of R-modules for $R = \mathbb{Z}$ and $R = F[x]$, with F a field.

(a) Find a similar characterization of R-modules for $R = \mathbb{Z}/n\mathbb{Z}$;

(b) Let $R = S[x]$ for some commutative ring S with 1. Prove that there is a natural correspondence between R-modules and pairs (M, T) where M is an S-module and $T : M \to M$ is an S-linear map.

(c) Let $R = F[x, y]$, with F a field. Use (b) to construct a natural correspondence between R-modules and triples (V, A, B) where V is an F-vector space and $A, B : V \to V$ are commuting linear transformations. **Note:** This can be done without (b), but (b) yields a complete proof which does not involve long and tedious verifications.

Problem 5 (practice): Let G be a group and $\mathbb{Z}[G]$ its integral group ring (see online Lecture 1 or DF, § 7.2 for definition). Let M be an abelian group. Show that there is a natural bijection between $\mathbb{Z}[G]$-module structures on M and actions of G on M by group automorphisms (that is, actions of G on M such that for any $g \in G$ the map $m \mapsto gm$ is an automorphism of the abelian group M).

Problem 6: An R-module M is called *simple (or irreducible)* if M has no submodules besides $\{0\}$ and M. An R-module M is called *indecomposable* if M is not isomorphic to $N \oplus P$ for nonzero R-modules N and P.

(a) Prove that every simple module is indecomposable

(b) Describe all simple $\mathbb{Z}$-modules and all finitely generated indecomposable $\mathbb{Z}$-modules. Deduce that an indecomposable module need not be simple.

Problem 7: An R-module M is called *cyclic* if M is generated (as an R-module) by one element.

(a) Prove that cyclic R-modules are precisely the ones which are isomorphic to R/I for some left ideal I of R.

(b) Prove that every simple module is cyclic. Then show that simple R-modules are precisely the ones which are isomorphic to R/I for some maximal left ideal I of R.

Problem 8. Let R be a commutative domain, and let I be a non-principal ideal of R. Prove that I, considered as an R-module (with left-multiplication action) is indecomposable but not cyclic. **Hint:** One way to prove that I is indecomposable is to show that any two elements of I are linearly dependent over R. **Note:** As we will prove in a couple of weeks, if R is a principal ideal domain, every finitely generated indecomposable module is cyclic.

Problem 9. Prove Schur's lemma [DF, problem 11, p.356].