6. Tensor, symmetric and exterior algebras

6.1. **The tensor algebra of a module.** Let M be an R-module. For each $k \geq 1$ let $T^k(M) = \underbrace{M \otimes_R \ldots \otimes_R M}_{k \text{ times}}$ and set $T^0(M) = R$. Define

$$T(M) = R \oplus T^1(M) \oplus T^2(M) \oplus \ldots = \oplus_{k=0}^{\infty} T^k(M)$$

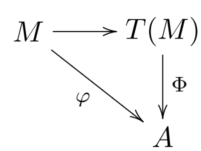
We already have an R-module structure on $T(M)$. Define multiplication on $T(M)$ by setting

$$(m_1 \otimes \ldots \otimes m_i) \cdot (m_1' \otimes \ldots \otimes m_j') = m_1 \otimes \ldots \otimes m_i \otimes m_1' \otimes \ldots \otimes m_j'$$

and extending to arbitrary elements of $T(M)$ by R-bilinearity.

That this multiplication is well defined can be proved similarly to what we did in the case of tensor products of algebra. It is easy to see that $T(M)$ becomes an R-algebra. Furthermore, by construction $T^i(M) \cdot T^j(M) \subseteq T^{i+j}(M)$, so $T(M) = \oplus_{k=0}^{\infty} T^k(M)$ is actually a graded R-algebra. The algebra $T(M)$ is called the tensor algebra of M.

Proposition 6.1 (Universal property of tensor algebras)**.** *Let M be an R-module and A an R-algebra. Then for any R-module homomorphism $\varphi : M \to A$ there exists unique R-algebra homomorphism $\Phi : T(M) \to A$ s.t. $\Phi_{|M} = \varphi$.*



Proof. Exercise (or see [DF,Theorem 31 on p.442]). □

Proposition 6.2. *Let M be a free R-module of rank n with basis $e_1, \ldots, e_n$. Then*

(a) *For any $k \geq 1$, $T^k(M)$ is a free R-module of rank n^k, and simple tensors $e_{i_1} \otimes \ldots \otimes e_{i_k}$ form a basis of $T^k(M)$.*
(b) *$T(M)$ is isomorphic to $R\langle x_1, \ldots, x_n\rangle$ (polynomials in non-commuting variables) as graded R-algebras.*

Proof. (a) By Example 4.3 if M is a free R-module with basis $e_1, \ldots, e_n$ and N is a free R-module with basis $f_1, \ldots, f_t$, then $M \otimes_R N$ is a free R-module with basis $\{e_i \otimes f_j\}$. The assertion of (a) follows from this by induction.
(b) Define $\Phi : R\langle x_1, \ldots, x_n\rangle \to T(M)$ by setting

$$\Phi\left(\sum r_{(i_1,\ldots,i_k)} x_{i_1} \ldots x_{i_k}\right) = \sum r_{(i_1,\ldots,i_k)} e_{i_1} \otimes \ldots \otimes e_{i_k}.$$

Then Φ is an homomorphism of graded R-algebras (by definition of the algebra structure on $T(M)$), and part (a) implies that Φ is bijective. $\square$

6.2. **Symmetric and exterior algebras.**

Definition. Let M be an R-module, $T(M)$ its tensor algebra and $C(M)$ the ideal of $T(M)$ generated by elements of the form $m_1 \otimes m_2 - m_2 \otimes m_1$ for $m_1, m_2 \in M$. The quotient algebra $S(M) = T(M)/C(M)$ is called the symmetric algebra of M.

Remark: 1. For each $k \geq 1$ let $S^k(M)$ be the image of $T^k(M)$ in $S(M)$. The ideal $C(M)$ is graded since it is generated by homogeneous elements, and therefore by Proposition 5.2 $S(M)$ is also a graded algebra with grading $S(M) = \oplus_{k=0}^{\infty} S^k(M)$.

2. $S(M)$ is a commutative R-algebra. Indeed, $T(M)$ is generated as a ring by $T^0(M) = R$ and $T^1(M) = M$, and therefore $S(M)$ is generated as a ring by $S^0(M)$ and $S^1(M)$. Note that $S^0(M)$ lies in the center of $S(M)$ (for $T^0(M)$ lies in the center of $T(M)$), and by construction any two elements of $S^1(M)$ commute. Thus, $S(M)$ is generated by a set of pairwise commuting elements, and therefore $S(M)$ is commutative.

Proposition 6.3. *Let M be a free R-module of rank n with basis $e_1, \ldots, e_n$. Then $S(M)$ is isomorphic to $R[x_1, \ldots, x_n]$ (polynomials in commuting variables) as graded R-algebras.*

Proof. Exercise. $\square$

Definition. Let M be an R-module, $T(M)$ its tensor algebra and $A(M)$ the ideal of $T(M)$ generated by elements of the form $m \otimes m$ for $m \in M$. The quotient algebra $\bigwedge(M) = T(M)/A(M)$ is called the exterior algebra of M.

The product in $\bigwedge(M)$ is denoted by the symbol $\wedge$. Thus given elements $m_1, \ldots, m_k \in M$ we denote by $m_1 \wedge \ldots \wedge m_k$ the image of $m_1 \otimes \ldots \otimes m_k$ in $\bigwedge(M)$.

Similarly to the case of symmetric algebras, the exterior algebra $\bigwedge(M)$ has a natural grading $\bigwedge(M) = \oplus_{k=0}^{\infty} \bigwedge^k(M)$ where $\bigwedge^k(M)$ is the image of $T^k(M)$ in $\bigwedge(M)$.

Proposition 6.4. *Let M be a free R-module of rank n with basis $e_1, \ldots, e_n$. The following hold:*

(i) *$\bigwedge(M)$ is isomorphic as an R-algebra to $R\langle x_1, \ldots, x_n \rangle / I$ where I is the ideal generated by $\{x_i^2, x_i x_j + x_j x_i\}$. Therefore, $\bigwedge(M)$ has the* *following presentation by generators and relations in the category of R-algebras:*

$$\bigwedge(M) = \langle e_1, \ldots, e_n \mid e_i \wedge e_i = 0 \text{ and } e_i \wedge e_j = -e_j \wedge e_i \text{ for } 1 \leq i, j \leq n \rangle.$$

(ii) *Let $k \geq 1$. Then $\bigwedge^k(M)$ is a free R-module with basis $\{e_{i_1} \wedge \ldots \wedge e_{i_k}\}$ where $i_1 < \ldots < i_k$. In particular, $rk(\bigwedge^k(M)) = \binom{n}{k}$.*

(iii) *$\bigwedge(M)$ is a finitely generated free R-module of rank 2^n.*

Proof. For (ii) see [DF,Corollary 37, p. 449], and (iii) follows from (ii) since $\sum_{k \geq 0} \binom{n}{k} = 2^n$. Let us prove (i).
Let $\Phi : R\langle x_1, \ldots, x_n \rangle \to T(M)$ be the isomorphism from Proposition 6.2. Then $\bigwedge(M) \cong R\langle x_1, \ldots, x_n \rangle / J$ where $J = \Phi^{-1}(A(M))$ is the image of $A(M)$ under Φ^{-1}. It is clear from the definition of Φ that J is the ideal generated by $\{(r_1 x_1 + \ldots + r_n x_n)^2 : r_i \in R\}$. We claim that $J = I$, for which it suffices to show that I contains generators of J and J contains generators of I.
The former is clear since

$$(r_1 x_1 + \ldots + r_n x_n)^2 = \sum_i r_i^2 x_i^2 + \sum_{i<j} r_i r_j (x_i x_j + x_j x_i) \in I$$

(and thus $J \subseteq I$). The reverse inclusion follows from the observation that $x_i x_j + x_j x_i = (x_i + x_j)^2 - x_i^2 - x_j^2 \in J$.

This proves the first assertion of (i), and the second assertion (regarding the presentation by generators and relations) is simply a restatement of the first one. □

6.3. **An interesting property of exterior algebras.** Let M and N be R-modules and $\varphi : M \to N$ an R-module homomorphism. By Proposition 6.1 φ yields a graded R-algebra homomorphism $\Phi : T(M) \to T(N)$ such that $\Phi(m_1 \otimes \ldots \otimes m_k) = \varphi(m_1) \otimes \ldots \varphi(m_k)$. It is easy to see that Φ maps $C(M)$ to $C(N)$ and $A(M)$ to $A(N)$, and thus Φ induces graded R-algebra homomorphisms $\Phi_{sym} : S(M) \to S(N)$ and $\Phi_{ext} : \bigwedge(M) \to \bigwedge(N)$

Now assume that R is a field, M is vector space over R of finite dimension n and $N = M$, so $\varphi : M \to M$ is an R-linear transformation. For each k we can restrict Φ_{ext} to $\bigwedge^k(M)$ to get an R-linear transformation

$$\Phi_{ext,k} : \bigwedge^k(M) \to \bigwedge^k(M).$$

Consider the case $k = n = \dim M$. Since $\dim \bigwedge^n(M) = \binom{n}{n} = 1$ by Proposition 6.4(ii), the map $\Phi_{ext,n}$ is just multiplication by some scalar $r(\Phi) \in R$.

Proposition 6.5. *The scalar $r(\Phi)$ is equal to* $\det \varphi$.

Proof. Let us see what happens when $n = 2$. Let $\{e_1, e_2\}$ be a basis of M. Then by Proposition 6.4(b) $\bigwedge^2(M)$ is R-spanned by the element $e_1 \wedge e_2$.

Now let $\varphi : M \to M$ be a linear transformation, and let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the matrix of φ with respect to $\{e_1, e_2\}$, so that $\varphi(e_1) = ae_1 + ce_2$ and $\varphi(e_2) = be_1 + de_2$.

Then by our construction $\Phi_{ext,2}(e_1 \wedge e_2) = (ae_1 + ce_2) \wedge (be_1 + de_2)$, and using anticommutativity and distributivity we get

$$(ae_1 + ce_2) \wedge (be_1 + de_2) = abe_1 \wedge e_1 + ade_1 \wedge e_2 + cbe_2 \wedge e_1 + cde_2 \wedge e_2 = \\ (ab - cd)e_1 \wedge e_2 = \det(\varphi)e_1 \wedge e_2.$$

Generalization of this proof to arbitrary n is a homework problem. □