5. Algebras over commutative rings

5.1. Two definition of R-algebras.

Definition 1. Let R be a commutative ring. An R-algebra is a ring A (with 1) together with a ring homomorphism $f : R \to A$ such that

(i) $f(1_R) = 1_A$;
(ii) $f(R) \subseteq Z(A)$, where $Z(A)$ is the center of A.

The pair (A, f) will also be called an R-algebra.

Example 5.1 (main): Let A be any ring and R a subring of $Z(A)$. Then (A, ι) is an R-algebra, where $\iota : R \to A$ is the inclusion mapping.

Example 5.2: Any ring A is a $\mathbb{Z}$-algebra (in a unique way). The map $f : \mathbb{Z} \to A$ is given by $f(n) = n_A$ where $n_A = \underbrace{1 + \ldots + 1}_{n \text{ times}}$.

Definition 2. Let R be a commutative ring. An R-algebra is a ring A which is also an R-module such that the multiplication map $A \times A \to A$ is R-bilinear, that is,

$$r * (ab) = (r * a) \cdot b = a \cdot (r * b) \text{ for any } a, b \in A, r \in R, \qquad (***)$$

where $*$ denotes the R-action on A.

Theorem 5.1. *Definitions 1 and 2 are equivalent. More precisely, given a commutative ring R and a ring A there is a natural bijection between R-algebra structures on A according to Definition 1 and R-algebra structures on A according to Definition 2.*

Proof. "$\longrightarrow$" Let $*$ be an R-module structure on A. Define $f : R \to A$ by $f(r) = r * 1_A$. Then

(i) f is a ring homomorphism since

$$f(r)f(s) = (r*1_A)\cdot(s*1_A) = 1_A\cdot(r*(s*1_A)) = 1_A\cdot(rs*1_A) = 1_A\cdot f(rs) = f(rs)$$

(ii) $f(1_R) = 1_R * 1_A = 1_A$ by module axioms
(iii) $f(R) \subset Z(A)$: for any $a \in A$ and $r \in R$ we have

$$a \cdot f(r) = a \cdot (r * 1_A) = r * (a \cdot 1_A) = r * (1_A \cdot a) = (r * 1_A) \cdot a = f(r) \cdot a.$$

"$\longleftarrow$" Suppose that $f : R \to A$ is a ring homomomorphism such that $f(1_R) = 1_A$ and $f(R) \subseteq Z(A)$. Define the R-action on A by $r * a = f(r) \cdot a$.

Verification of R-module axioms is straightforward. Let us check that multiplication on A is R-bilinear:

$$r * (ab) = f(r)ab$$
$$(r * a) \cdot b = (f(r)a) \cdot b = f(r)ab$$
$$a \cdot (r * b) = a \cdot (f(r)b) = f(r)ab \text{ since } f(r) \in Z(A).$$

Exercise: check that the correspondence we constructed is indeed bijective.

□

Definition. Let A and B be R-algebras. A map $f : A \to B$ is called an R-algebra homomorphism if

(i) f is a ring homomorphism and $f(1_A) = 1_B$.
(ii) f is an R-module homomorphism.

5.2. **Examples of R-algebras.**

Example 5.3: If A and B are R-algebras, then $A \oplus B$ is an R-algebra (in a natural way) – use Definition 2.

Example 5.4: If A is an R-algebra and I an ideal of A, then A/I is an R-algebra (in a natural way) – use Definition 1.

Example 5.5: Let $X = \{x_1, \dots, x_n\}$ be a finite set and $A = R\langle x_1, \dots, x_n\rangle$ the ring of polynomials over R in non-commuting variables $x_1, \dots, x_n$. Then A is a free R-algebra on X in the following sense: for any R-algebra S any map $f : X \to S$ uniquely extends to an R-algebra homomorphism $f_* : A \to S$.

Example 5.6: Let $X = \{x_1, \dots, x_n\}$ be a finite set and $A = R[x_1, \dots, x_n]$ the ring of polynomials over R in commuting variables $x_1, \dots, x_n$. Then A is a free commutative R-algebra on X. Exercise: formulate the corresponding universal property.

5.3. **Tensor product of algebras.** Let A and B be two R-algebras. We have already defined the R-module $A \otimes_R B$. To turn $A \otimes_R B$ into R-algebra we define muliplication on $A \otimes_R B$ by setting $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = a_1a_2 \otimes b_1b_2$ and extending to arbitrary elements of $A \otimes_R B$ by distributivity.

Things to check:

(1) multiplication on $A \otimes_R B$ is well defined
(2) multiplication on $A \otimes_R B$ is bilinear

(2) is an easy exercise. (1) can be done going back to the definition of tensor products, but this is boring. An elegant way to prove this is as follows:
Define the multilinear map $m : A \times B \times A \times B \to A \otimes B$ by $m(a_1, b_1, a_2, b_2) = a_1a_2 \otimes b_1b_2$. Then m is clearly R-multilinear. Thus by Theorem 4.2' there

exists an R-module homomorphism $\mu : A \otimes B \otimes A \otimes B \to A \otimes B$ such that $\mu(a_1 \otimes b_1 \otimes a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$.
Since $A \otimes B \otimes A \otimes B$ is naturally isomorphic to $(A \otimes B) \otimes (A \otimes B)$, the map μ yields an R-blinear map $\widetilde{m} : (A \otimes B) \times (A \otimes B) \to A \otimes B$ such that $\widetilde{m}(a_1 \otimes b_1, a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$. Then $\widetilde{m}$ is the multiplcation map whose existence we wanted to show.

5.4. **Graded algebras.**

Definition. (a) Let A be an R-algebra. A grading of A is a collection of R-submodules $\{A_n\}_{n=0}^{\infty}$ such that $A = \oplus_{n=0}^{\infty} A_n$ and $A_n \cdot A_m \subseteq A_{m+n}$ for all m, n.
(b) A graded R-algebra is an R-algebra with a chosen grading.

Example 5.7: Let $A = R[x_1, \ldots, x_k]$. Then A has a natural grading where

$$A_n = \{\text{homogeneous polynomials of degree } n\} \cup \{0\}.$$

The algebra $R\langle x_1, \ldots, x_k\rangle$ admits analogous grading.

Definition. Let $A = \oplus_{n=0}^{\infty} A_n$ be a graded R-algebra. An ideal I of A is called graded if $I = \oplus_{n=0}^{\infty}(I \cap A_n)$

Example 5.8: (a) Let $A = R[x]$ and $I = (x+1)$. Then I is not graded. Indeed, it is easy to see that I contains no monomials, and therefore

$$\oplus_{n=0}^{\infty}(I \cap R[x]_n) = \{0\} \neq I.$$

(b) Let $A = \oplus_{n=0}^{\infty} A_n$ be any graded R-algebra and I an ideal of A, generated by homogeneous elements. Then I is graded.

Proposition 5.2. *Let $A = \oplus_{n=0}^{\infty} A_n$ be a graded R-algebra and I a graded ideal of A. Let $(A/I)_n = (A_n + I)/I$ be the image of A_n in A/I. Then $A/I = \oplus_{n=0}^{\infty}(A/I)_n$, so A/I is also a graded R-algebra in a natural way. Note that $(A/I)_n \cong A_n / I \cap A_n$ as R-modules.*

Proof. Exercise or see [DF, Proposition 33] on p. 444. □

Definition. Let $A = \oplus_{n=0}^{\infty} A_n$ and $B = \oplus_{n=0}^{\infty} B_n$ be R-algebras. A map $\varphi : A \to B$ is called a homomorphism of graded R-algebras if φ is a homomorphism of R-algebras and φ respects the grading, that is, $\varphi(A_n) \subseteq B_n$ for each n.