Today: R denotes a commutative ring.

4. Tensor products and bilinear maps

Definition. Let M and N be R-modules and L an abelian group.

(a) A map $\varphi : M \times N \to L$ is called R-balanced if
  (i) $\varphi(m_1 + m_2, n) = \varphi(m_1, n) + \varphi(m_2, n)$
  (ii) $\varphi(m, n_1 + n_2) = \varphi(m, n_1) + \varphi(m, n_2)$
  (iii) $\varphi(m, rn) = \varphi(rm, n)$ for any $r \in R$, $m \in M$, $n \in N$

(b) Now suppose that L is also an R-module. Then a map $\varphi : M \times N \to L$ is called R-bilinear if φ is R-balanced and $\varphi(m, rn) = r\varphi(m, n)$.

Example 4.1: The map $\iota : M \times N \to M \otimes_R N$ given by $\iota(m, n) = m \otimes n$ is R-bilinear. This follows from defining relations of tensor products.

Example 4.2: Suppose M and N are finitely generated free R-modules. Let $\{x_1, \ldots, x_k\}$ be a basis of M and $\{y_1, \ldots, y_t\}$ be a basis of N.

Let L be another R-module, and choose arbitrary elements $l_{ij} \in L$, with $1 \le i \le k$ and $1 \le j \le t$. Then there exists unique R-bilinear map $\varphi : M \times N \to L$ such that $\varphi(x_i, y_j) = l_{ij}$. In fact, φ is given by the formula

$$\varphi(\sum r_i x_i, \sum s_j y_j) = \sum_{i,j} r_i s_j l_{ij}.$$

Theorem 4.1. *Let M and N be R-modules and L an abelian group. Then there is a bijection*

$$\Omega = \left\{ \begin{array}{l} R\text{-balanced maps} \\ \varphi : M \times N \to L \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{group homomorphisms} \\ \Phi : M \otimes_R N \to L \end{array} \right\} = \Delta$$

which maps an R-balanced map $\varphi \in \Omega$ to a group homomorphism $\Phi \in \Delta$ such that $\Phi(m \otimes n) = \varphi(m, n)$ for any $m \in M$, $n \in N$

Proof. "$\longrightarrow$" (a map $f : \Omega \to \Delta$). Recall that $M \otimes_R N = F/I$ where F is the free abelian group on $M \times N$ and I is the subgroup generated by $\{(m, n_1 + n_2) - (m, n_1) - (m, n_2) \text{ etc. }\}$

Now let $\varphi : M \times N \to L$ be R-balanced. Since F is a free $\mathbb{Z}$-module on $M \times N$, there is a group homomorphism $\widetilde{\Phi} : F \to L$ such that $\widetilde{\Phi}((m, n)) = \varphi(m, n)$. Then $I \subset \operatorname{Ker} \widetilde{\Phi}$ precisely because φ is R-balanced. For instance,

$$\begin{aligned} \widetilde{\Phi}((m, n_1 + n_2) - (m, n_1) - (m, n_2)) &= \widetilde{\Phi}((m, n_1 + n_2)) - \widetilde{\Phi}((m, n_1)) - \widetilde{\Phi}((m, n_2))) \\ &= \varphi(m, n_1 + n_2) - \varphi(m, n_1) - \varphi(m, n_2) = 0, \end{aligned}$$

where the first equality holds since $\widetilde{\Phi}$ is a group homomorphism and the last equality holds since φ is R-balanced.
Thus, $\widetilde{\Phi}$ induces a group homomorhism $\Phi : M \otimes_R N = F/I \to L$ such that $\Phi(m \otimes n) = \varphi(m, n)$. We set $f(\varphi) = \Phi$.

"$\longleftarrow$" (a map from $g : \Delta \to \Omega$). This is easy – just set

$$(g(\Phi))(m, n) = \Phi(m \otimes n).$$

Then $g(\Phi)$ is R-balanced by defining relations in $M \otimes_R N$.

Thus, we defined two maps $f : \Omega \to \Delta$ and $g : \Delta \to \Omega$, and it remains to check that f and g are mutually inverse.

By construction for any $\varphi \in \Omega$ we have

$$g(f(\varphi))(m, n) = f(\varphi)(m \otimes n) = \varphi(m, n).$$

Thus $g \circ f = id_\Omega$. Similarly, $f(g(\Phi))(m \otimes n) = \Phi(m \otimes n)$. Since a group homomorphism $M \otimes_R N \to L$ is uniquely determined by its values on simple tensors, we must have $f \circ g = id_\Delta$ □

Here is the variation of Theorem 4.1 dealing with bilinear maps.

Theorem 4.2. *Let M and N be R-modules and L another R-module. Then there is a bijection*

$$\Omega = \left\{ \begin{array}{l} R\text{-bilinear maps} \\ \varphi : M \times N \to L \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} R\text{-module homomorphisms} \\ \Phi : M \otimes_R N \to L \end{array} \right\} = \Delta$$

which maps a R-bilinear map $\varphi \in \Omega$ to an R-module homomorphism $\Phi \in \Delta$. such that $\Phi(m \otimes n) = \varphi(m, n)$ for any $m \in M$, $n \in N$

Proof. Very similar to that of Theorem 4.1. □

4.1. **Applications of Theorem 4.2.**

Example 4.3: Let M, N be finitely generated free R-modules, $X = \{x_1, \ldots, x_k\}$ a basis of M and $Y = \{y_1, \ldots, y_t\}$ a basis of N. Then $\{x_i \otimes y_j\}$ is a basis of $M \otimes_R N$.

Remark: Finite generation assumption is not essential.

Proof. We know from Example 3.2 that $\{x_i \otimes y_j\}$ generates $M \otimes_R N$, so we only need to check linear independence. Suppose that $\sum_{i,j} r_{ij} x_i \otimes y_j = 0$ where $r_{ij} \in R$ and not all r_{ij} are zero.

WOLOG $r_{11} \neq 0$. By Example 4.2 there exists an R-bilinear map $\varphi : M \times N \to R$ such that $\varphi((x_1, y_1)) = 1$ and $\varphi((x_i, y_j)) = 0$ if $(i, j) \neq (1, 1)$. By Theorem 4.2 there is an R-module homomorphism $\Phi : M \otimes N \to R$ such that $\Phi(x_i \otimes y_j) = \varphi((x_i, y_j))$. Then

$$\Phi(\sum_{i,j} r_{ij} x_i \otimes y_j) = \sum_{i,j} r_{ij} \varphi(x_i, y_j) = r_{11} \neq 0,$$

which is a contradiction. □

Example 4.4: Prove that for any R-module M we have $R \otimes_R M \cong M$ (as R-modules).

Claim. *Any element of $R \otimes_R M$ is equal to $1 \otimes m$ for some $m \in M$.*

Proof of the claim. Any element of $R \otimes_R M$ can be written as

$$\sum r_i \otimes m_i = \sum (1 \cdot r_i) \otimes m_i = \sum 1 \otimes r_i m_i = 1 \otimes \sum r_i m_i.$$

□

Proof of Example 4.4: Define the map $\Phi : M \to R \otimes_R M$ by $\Phi(m) = 1 \otimes m$. Clearly, Φ is an R-module homomorphism, and Φ is surjective by the Claim. To prove that Φ is injective it is enough to show that $1 \otimes m \neq 0$ for $m \neq 0$. By Theorem 4.2 it is enough to find an R-bilinear map $\varphi : R \times M \to M$ such that $\varphi(1, m) \neq 0$. The map $\varphi(r, m) = rm$ has such property. □

Proposition 4.3. *Let M, N, P be R-modules. Then there exist natural R-module isomorphisms*

(i) $M \otimes N \cong N \otimes M$;
(ii) $(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$
(iii) $(M \oplus N) \otimes P \cong (M \otimes P) \oplus (N \otimes P)$

Proof. We shall prove (i); see [DF] for (ii) and (iii). Consider the map $\varphi : M \times N \to N \otimes M$ given by $\varphi(m, n) = n \otimes m$. The map φ is clearly R-bilinear, and thus there exists an R-module homomorphism $f : M \otimes N \to N \otimes M$ such that $f(m \otimes n) = n \otimes m$ for each $m \in M$, $n \in N$. Similarly, there is an R-module homomorphism $g : N \otimes M \to M \otimes N$ such that $g(n \otimes m) = m \otimes n$. The composition $gf : M \otimes N \to M \otimes N$ is an R-module homomorphism, which fixes all simple tensors and hence fixes everything. So $gf = id_M$, and similarly $fg = id_N$ □

4.2. **Generalizations of Theorems 4.1 and 4.2.** Theorems 4.1 and 4.2 have natural generalizations dealing with multilinear maps. For instance, here is the generalization of Theorem 4.2.

Definition. Let $k \geq 2$ and let $M_1, \ldots, M_k$ and L be R-modules. A map $\varphi : M_1 \times \ldots \times M_k \to L$ is called R-multilinear if for any $1 \leq i \leq k$ we have

$$\varphi(m_1, \ldots, m_{i-1}, m_i + rm_i', m_{i+1}, \ldots, m_k) =$$
$$\varphi(m_1, \ldots, m_{i-1}, m_i, m_{i+1}, \ldots, m_k) + r\varphi(m_1, \ldots, m_{i-1}, m_i', m_{i+1}, \ldots, m_k)$$

for all $m_j \in M_j, 1 \leq j \leq k, m_i' \in M_i$ and $r \in R$.

Theorem 4.2'. *Then there is a bijection* $\varphi \leftrightarrow \Phi$ *between*

$$\left\{ \begin{array}{l} R\text{-}\textit{multilinear maps} \\ \varphi : M_1 \times M_2 \times \ldots \times M_k \to L \end{array} \right\} \textit{ and } \left\{ \begin{array}{l} R\text{-}\textit{module homomorphisms} \\ \Phi : M_1 \otimes M_2 \otimes \ldots \otimes M_k \to L \end{array} \right\}$$

s.t. $\Phi(m_1 \otimes m_2 \otimes \ldots \otimes m_k) = \varphi(m_1, m_2, \ldots, m_k)$ *for all* $m_i \in M_i, 1 \leq i \leq k$.