Addendum to Lecture 2.

Definition. Let R be a ring and X a set. **The** free R-module on X is the set of formal linear combinations $\sum_{x \in X} r_x x$ where $r_x \in R$ and only finitely many r_x are nonzero. We will denote it by $F_R(X)$.

Clearly, X is a basis of $F_R(X)$, so $F_R(X)$ is free according to definition from Lecture 2.

3. Tensor products of modules

In this lecture we shall define two types of tensor products of modules. In fact, both types are special cases of a more general construction, but they are usually applied for different purposes.

3.1. **Basic motivation (vector space case).**

1. Let $F \subset K$ be two fields and V an F-vector space. We want to define a K-vector space $K \otimes_F V$ which consists of "linear combinations of elements of V with coefficients from K".
2. Suppose F is a field, V, W vector spaces over F. Then

$$\dim(V \oplus W) = \dim(V) + \dim(W).$$

We want to define another F-vector space $V \otimes_F W$ s.t.

$$\dim(V \otimes_F W) = \dim(V) \cdot \dim(W).$$

One way to do it is the following – pick a basis $\{e_\alpha\}$ of V, a basis $\{f_\beta\}$ of W, and let $V \otimes_F W$ be the set of formal linear combinations of symbols $\{e_\alpha \otimes f_\beta\}$ with coefficients from F.

We are looking for the definition of $V \otimes_F W$ which does not involve a choice of bases and also generalizes to modules over rings which are not fields.

3.2. **Tensor products of type I (extension of scalars).**

Suppose we are given two rings R and S, with $R \subseteq S$ and an R-module M. We shall define certain S-module $S \otimes_R M$.

First we define $S \otimes_R M$ as an abelian group. Let $X = S \times M$ be the set of pairs $\{(s, m) : s \in S, m \in M\}$, and let F be the free abelian group on X (= free $\mathbb{Z}$-module on X).

Define $S \otimes_R M$ to be the quotient F/I where I is the subgroup of F generated by the elements

$$\begin{array}{ll} (s_1+s_2,m)-(s_1,m)-(s_2,m) & s,s_1,s_2 \in S, \\ (s,m_1+m_2)-(s,m_1)-(s,m_2) & m,m_1,m_2 \in M \\ (sr,m)-(s,rm) & r \in R \end{array}$$

Denote the image of (s,m) in $F/I = S \otimes_R M$ by $s \otimes m$.

Thus, $S \otimes_R M$ can also be defined as an abelian group with generators $\{s \otimes m, s \in S, m \in M\}$ and relations

$$(s_1+s_2)\otimes m = s_1\otimes m + s_2 \otimes m;\ s\otimes(m_1+m_2) = s\otimes m_1 + s\otimes m_2;\ (sr)\otimes m = s\otimes rm$$

Every element of $S \otimes_R M$ can be written as a finite sum $\sum s_i \otimes m_i$ (such representation is NOT unique).

Informally, we can think of $s \otimes m$ as "scalar s times vector m."

Now we define the S-module structure on $S \otimes_R M$. Note that

(1) First note that the abelian group F has a natural structure of an S-module where S-action (denoted by $*$ below) is given by

$$s * \sum \pm(s_i, m_i) = \sum \pm(ss_i, m_i).$$

(2) The subgroup I defined above is clearly an S-submodule

Thus, $S \otimes_R M = F/I$ can be given the structure of a quotient S-module. The action of S on $S \otimes_R M$ is explicitly given by

$$s(\sum s_i \otimes m_i) = \sum ss_i \otimes m_i. \qquad (***)$$

Remark: We can use (***) as the definition of S-action, but then we would have to prove that this action is well defined.

3.3. **Tensor products of type II (regular tensor products).** Let R be a commutative ring, and let M and N be R-modules. Define the R-module $M \otimes_R N$ as follows.

Let $X = \{(m,n) : m \in M, n \in N\}$, F the free abelian group on X and I the subgroup of I generated by the elements

$$\begin{array}{ll} (m_1+m_2,n)-(m_1,n)-(m_2,n) & m,m_1,m_2 \in M, \\ (m,n_1+n_2)-(m,n_1)-(m,n_2) & n,n_1,n_2 \in N \\ (rm_1,m_2)-(m_1,rm_2) & r \in R \end{array}$$

Define $M \otimes_R N = F/I$, and set $m \otimes n$ be the image of (m,n) in $M \otimes_R N$. Finally, turn $M \otimes_R N$ into an R-module by setting

$$r(\sum m_i \otimes n_i) = rm_i \otimes n_i.$$

This action is well defined by the same argument as with type I tensor products.

Definition. Elements of $M \otimes_R N$ of the form $m \otimes n$ are called simple tensors.

As with type I tensor products we have the following relations between simple tensors (in fact, these are defining relations):

$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n \qquad m, m_1, m_2 \in M,$$
$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2 \qquad n, n_1, n_2 \in N$$
$$(rm_1) \otimes m_2 = m_1 \otimes rm_2 = r(m_1 \otimes m_2) \qquad r \in R$$

Remember this:

(i) Every element of $M \otimes_R N$ can be written as $\sum_{i=1}^{k} m_i \otimes n_i$. In particular, $M \otimes_R N$ is generated by the simple tensors.
(ii) In most cases, NOT every element of $M \otimes_R N$ is a simple tensor.

Remark: 1. If R is not commutative, one can still define type II tensor product $M \otimes_R N$, but this time we have to assume that M is a right R-module and N is a left R-module. Of course, when we define the subgroup I, the third family of elements changes to $\{(m_1 r, m_2) - (m_1, rm_2)\}$.
2. If S is a ring containing R, we can consider S as a right R-module with action given by right multiplication. Thus, if M is a left R-module, we can define type II tensor product $S \otimes_R M$ (which is an R-module).

We also have type I tensor product $S \otimes_R M$ which is an S-module and hence also an R-module. Fortunately, this does not cause a confusion since in both cases we get the same R-module (which is easily seen from definitions).

3.4. **Computing in tensor products.**
Example 3.1: Let $k \in \mathbb{Z}$, $k \geq 2$. Prove that $\mathbb{Z}/k\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = \{0\}$.
Proof: For each $x \in \mathbb{Z}/k\mathbb{Z}$ and $y \in \mathbb{Q}$ we have

$$x \otimes y = x \otimes (k \cdot \frac{y}{k}) = kx \otimes \frac{y}{k} = 0 \otimes \frac{y}{k} = 0.$$

Since $\mathbb{Z}/k\mathbb{Z} \otimes \mathbb{Q}$ is generated by simple tensors $x \otimes y$, we get $\mathbb{Z}/k\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = \{0\}$.

Example 3.2: Let R be a commutative ring, and let M and N be R-modules. Let X be a generating set of M and Y a generating set of N. Prove that $Z = \{x \otimes y : x \in X, y \in Y\}$ is a generating set of $M \otimes N$.
Proof: Let $L = RZ$ be the submodule generated by Z. It is enough to show that L contains every simple tensor $m \otimes n$.
Take any $m \in M, n \in N$. By assumption we can write $m = \sum r_i x_i$ and $n = \sum s_i y_i$ (both sums are finite) where $x_i \in X$, $y_i \in Y$ and $r_i, s_i \in R$.

Then

$$m \otimes n = (\sum r_i x_i) \otimes (\sum s_j y_j) = \sum_{i,j} (r_i x_i) \otimes (s_j y_j) =$$
$$\sum_{i,j} (r_i s_j x_i) \otimes y_j = \sum_{i,j} r_i s_j (x_i \otimes y_j) \in RZ = L.$$