26.1. **Direct limits.**

Definition. A poset A is called a <u>directed set</u> if for any $\alpha, \beta \in A$ there exists $\gamma \in A$ s.t. $\alpha \leq \gamma$ and $\beta \leq \gamma$.

Definition. Let $\mathcal{C}$ be a category. A <u>direct system in $\mathcal{C}$</u> consists of a directed set A, a collection of objects $\{X_\alpha\}_{\alpha \in A}$ of $\mathcal{C}$ and morphisms $\iota_{\alpha\beta} : X_\alpha \to X_\beta$ for any $\alpha \leq \beta$ s.t.

(i) $\iota_{\alpha\alpha} = id_{X_\alpha}$ for all $\alpha \in A$
(ii) $\iota_{\beta\gamma} \circ \iota_{\alpha\beta} = \iota_{\alpha\gamma}$ whenever $\alpha \leq \beta \leq \gamma$.

Remark: The notions of a direct system and inverse system (defined below) make sense even if the poset A is not assumed to be directed. However many important results only hold when A is directed.

Definition. Let $\mathcal{C}$ be a category and $(A, \{X_\alpha\}, \{\iota_{\alpha\beta}\})$ a direct system in $\mathcal{C}$. An object $X \in Ob(\mathcal{C})$ is called a <u>direct limit</u> of this system if there exist morphisms $\iota_\alpha : X_\alpha \to X$ for $\alpha \in A$ with the following property:

(i) For any $\alpha \leq \beta$ the following diagram commutes:

$$\begin{array}{ccc} X_\alpha & & \\ {\scriptstyle \iota_{\alpha\beta}}\downarrow & \searrow^{\iota_\alpha} & \\ X_\beta & \xrightarrow[\iota_\beta]{} & X \end{array}$$

(ii) Given any $Y \in Ob(\mathcal{C})$ and morphisms $\varphi_\alpha : X_\alpha \to Y$ s.t. the diagram

$$\begin{array}{ccc} X_\alpha & & \\ {\scriptstyle \iota_{\alpha\beta}}\downarrow & \searrow^{\varphi_\alpha} & \\ X_\beta & \xrightarrow[\varphi_\beta]{} & X \end{array}$$

commutes for $\alpha \leq \beta$, there exists unique morphism $\varphi : X \to Y$ s.t. the following diagram commutes for all $\alpha \in A$:

$$\begin{array}{ccc} X_\alpha & \xrightarrow{\iota_\alpha} & X \\ & {\scriptstyle \varphi_\alpha}\searrow & \downarrow{\scriptstyle \varphi} \\ & & Y \end{array}$$

If a direct limit exists, it is unique up to $\mathcal{C}$-isomorphism and is denoted by $\varinjlim X_\alpha$.

<u>Examples:</u> 1. Let $\mathcal{C}$ be the category of sets. The simplest example of a direct system in $\mathcal{C}$ is a collection $\{X_\alpha\}$ of subsets of the same set Y which form a chain, where the maps $\iota_{\alpha\beta}$ are natural inclusions. In this case $\varinjlim X_\alpha = \cup X_\alpha$. The same holds in the categories of groups, abelian groups, rings etc.

2. Let $\mathcal{C}$ be the category of sets and $(A, \{X_\alpha\}, \{\iota_{\alpha\beta}\})$ an arbitrary direct system in $\mathcal{C}$. Define the relation $\sim$ on $\sqcup X_\alpha$ as follows: if $x \in X_\alpha$ and $y \in X_\beta$, then $x \sim y$ if there exists $k \in A$ s.t. $\iota_{\alpha\gamma}(x) = \iota_{\beta\gamma}(y)$ (here we identify each X_α with its image in $\sqcup X_\alpha$). Then $\sim$ is an equivalence relation (because A is a directed set), and one can show that

$$\varinjlim X_\alpha = \sqcup X_\alpha / \sim .$$

Remark: If A is not assumed to be directed, it is still true that $\varinjlim X_\alpha = \sqcup X_\alpha / \sim$ for certain equivalence relation, but the definition of $\sim$ is less explicit: one defines $\sim$ to be the smallest equivalent relation for which $x \sim \iota_{\alpha\beta}(x)$ for any $\alpha \leq \beta$ and $x \in X_\alpha$.

3. Let $\mathcal{C}$ be the category of abelian groups and $(A, \{X_\alpha\}, \{\iota_{\alpha\beta}\})$ an arbitrary direct system in $\mathcal{C}$. Then $\varinjlim X_\alpha = \oplus X_\alpha / I$ where I is the subgroup of $\oplus X_\alpha$ generated by the set

$$\{\iota_{\alpha\beta}(x) - x \text{ where } \alpha \leq \beta,\ x \in X_\alpha\}.$$

Here we do not need to assume that A is directed.

26.2. **Inverse limits.**

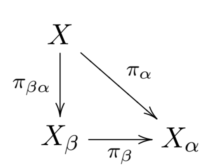
Definition. Let $\mathcal{C}$ be a category. An <u>inverse system in $\mathcal{C}$</u> consists of a directed set A, a collection of objects $\{X_\alpha\}_{\alpha \in A}$ of $\mathcal{C}$ and morphisms $\pi_{\beta\alpha} : X_\beta \to X_\alpha$ for any $\alpha \leq \beta$ s.t.

(i) $\pi_{\alpha\alpha} = id_{X_\alpha}$ for all $\alpha \in A$

(ii) $\pi_{\beta\alpha} \circ \pi_{\gamma\beta} = \pi_{\gamma\alpha}$ whenever $\alpha \leq \beta \leq \gamma$.

Definition. Let $\mathcal{C}$ be a category and $(A, \{X_\alpha\}, \{\pi_{\beta\alpha}\})$ an inverse system in $\mathcal{C}$. An object $X \in Ob(\mathcal{C})$ is called an <u>inverse limit</u> of this system if there exist morphisms $\pi_\alpha : X \to X_\alpha$ for $\alpha \in A$ with the following property:

(i) For any $\alpha \leq \beta$ the following diagram commutes:



(ii) Given any $Y \in Ob(\mathcal{C})$ and morphisms $\varphi_\alpha : Y \to X_\alpha$ s.t. the diagram

$$\begin{array}{ccc} Y & & \\ {\scriptstyle \pi_{\beta\alpha}}\big\downarrow & \searrow^{\varphi_\alpha} & \\ X_\beta & \xrightarrow[\varphi_\beta]{} & X_\alpha \end{array}$$

commutes for $\alpha \le \beta$, there exists unique morphism $\varphi : Y \to X$ s.t. the following diagram commutes for all $\alpha \in A$:

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ & {}_{\varphi_\alpha}\searrow & \big\downarrow {\scriptstyle \pi_\alpha} \\ & & X_\alpha \end{array}$$

If an inverse limit exists, it is unique up to $\mathcal{C}$-isomorphism and is denoted by $\varprojlim X_\alpha$.

Easy fact: Inverse limits always exist in the categories of sets, groups, rings etc. and admit the following description:

$$\varprojlim X_\alpha = \{(x_\alpha) \in \prod X_\alpha \text{ s.t. } \pi_{\beta\alpha}(x_\beta) = x_\alpha \text{ for all } \alpha \le \beta\}.$$

26.3. **Examples of inverse systems.**

1. Let R be a ring with 1 and I an ideal of R. For $n \in \mathbb{N}$ let $R_n = R/I^n$. Then $\{R_n\}_{n\in\mathbb{N}}$ is an inverse system where the maps $\pi_{mn} : R_m \to R_n$ are natural projections. Then $\varprojlim R_n = \widehat{R}_I$, the I-adic completion of R, as proved in Algebra-I.

2. Consider the following inverse system in the category of sets. The indexing set will be $\mathbb{N}$ (with the natural order), and each X_n is also taken to be $\mathbb{N}$. Define $\pi_{mn} : X_m \to X_n$ for $n \le m$ by $\pi_{mn}(x) = x + (m - n)$. Then it is easy to see that $\varprojlim X_n = \emptyset$.

Remark: If $\{X_\alpha\}$ is an inverse system of finite sets, then $\varprojlim X_n$ is always non-empty. This can be proved using Tychonoff's theorem (product of compact sets is compact). The fact that the indexing set A is directed is essential for this proof.

3. Let G be a group. Let $\mathfrak{A}$ be the set of all normal subgroups of finite index, ordered by reverse inclusion, that is, $K \le N$ if and only if $N \subseteq K$. Then $\mathfrak{A}$ is a directed set since if $K, N \in \mathfrak{A}$, then $K \cap N \in \mathfrak{A}$ as well. Consider the inverse system $\{G/N\}_{N\in\mathfrak{A}}$ where the maps $\pi_{K,N} : G/K \to G/N$ are natural projections. The inverse limit $\varprojlim_{N\in\mathfrak{A}} G/N$ is called the profinite completion of G and is commonly denoted by $\widehat{G}$.