25. Some category theory

25.1. **Categories.** A category $\mathcal{C}$ consists of the following data:

- objects $Ob(\mathcal{C})$
- for any $X, Y \in Ob(\mathcal{C})$ a set $Mor(X,Y) = Mor_{\mathcal{C}}(X,Y)$ called morphisms from X to Y
- for any triple $X, Y, Z \in Ob(\mathcal{C})$ a map

$$Mor(X,Y) \times Mor(Y,Z) \to Mor(X,Z)$$
$$(f,g) \mapsto g \circ f$$

  called the composition law of morphisms.

The following axioms must be satisfied:

(1) The sets $Mor(X,Y)$ and $Mor(X',Y')$ are disjoint unless $X = X'$ and $Y = Y'$.
(2) Composition of morphisms is associative, that is, for any $f \in Mor(X,Y)$, $g \in Mor(Y,Z)$ and $h \in Mor(Z,W)$ we have

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(3) For any $X \in Ob(\mathcal{C})$ there is identity morphism $1_X \in Mor(X,X)$ with the following property: if Y is any object of $\mathcal{C}$, then $f \circ 1_X = f$ for any $f \in Mor(X,Y)$ and $1_X \circ g = g$ for any $g \in Mor(Y,X)$.

Notation: We will often write $f : X \to Y$ instead of $f \in Mor(X,Y)$.

Here are some basic examples of categories.

Examples: (1) $\mathcal{C} = SET$, the category of sets. Objects of $\mathcal{C}$ are arbitrary sets and $Mor(X,Y) = Func(X,Y)$, all functions from X to Y. The composition of morphisms is the usual composition of functions.

(2) $\mathcal{C} = GRP$, the category of groups. Objects are all groups, $Mor(X,Y)$ is the set of groups homomorphisms from X to Y, the composition of morphisms is the usual composition of functions.

(3) $\mathcal{C} = TOP$, the category of topological spaces. Objects are topological spaces, $Mor(X,Y)$ is the set of continuous functions from X to Y, the composition of morphisms is the usual composition of functions.

Here is an example of rather different kind.

(4) Let A be a poset with partial order relation $\leq$. Then we can consider the following category $\mathcal{C}$. The objects of $\mathcal{C}$ are simply elements of A, and

morphisms are defined by setting

$$Mor(x,y) = \begin{cases} \emptyset & \text{if } x \not\leq y \\ \text{The one element set consisting of the pair } (x,y) & \text{if } x \leq y. \end{cases}$$

The composition of morphisms $Mor(x,y) \times Mor(y,z) \to Mor(x,z)$ is defined as follows:

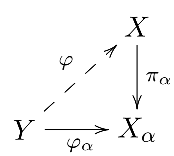
(i) If $x \not\leq y$ or $y \not\leq z$, then $Mor(x,y) \times Mor(y,z) = \emptyset$, so there exists unique map $Mor(x,y) \times Mor(y,z) \to Mor(x,z)$ (the "do nothing" map)

(ii) If $x \leq y$ and $y \leq z$, then $x \leq z$ by transitivity, so $|Mor(x,y)| = |Mor(y,z)| = |Mor(x,z)| = 1$. Again there exists unique map $Mor(x,y) \times Mor(y,z) \to Mor(x,z)$ given by $((x,y),(y,z)) \mapsto (x,z)$.

Finally, associativity of composition is automatic and identity morphisms 1_x exist since $x \leq x$.

25.2. **Products and coproducts.**

Definition. Let $\mathcal{C}$ be a category and $\{X_\alpha\}$ a collection of objects of $\mathcal{C}$. An object $X \in Ob(\mathcal{C})$ is called a product of $\{X_\alpha\}$ denoted $\prod_{\mathcal{C}} X_\alpha$ if there exist morphisms $\pi_\alpha : X \to X_\alpha$ for each α s.t. for any $Y \in Ob(\mathcal{C})$ and any morphisms $\varphi_\alpha : Y \to X_\alpha$ there is unique morphism $\varphi : Y \to X$ s.t. for each α we have $\varphi_\alpha = \pi_\alpha \varphi$, or equivalently, the following diagram is commutative:



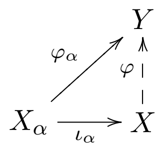
A standard argument shows that if a product $\prod_{\mathcal{C}} X_\alpha$ exists, it is unique up to $\mathcal{C}$-isomorphism; however, a product need not exist in general.

Examples: (1) Let $\mathcal{C}$ be the category of sets (resp. groups, abelian groups, rings). Then $\prod_{\mathcal{C}} X_\alpha$ always exists and coincides with the usual direct product of sets (resp. groups, abelian groups, rings).

(2) Let $\mathcal{C}$ be the category of fields (with morphisms being field embeddings). Then products in $\mathcal{C}$ do not always exist (in fact, almost never exist).

Coproducts are defined in the same way as products with all arrows reversed:

Definition. Let $\mathcal{C}$ be a category and $\{X_\alpha\}$ a collection of objects of $\mathcal{C}$. An object $X \in Ob(\mathcal{C})$ is called a coproduct of $\{X_\alpha\}$ denoted $\sqcup_{\mathcal{C}} X_\alpha$ if there exist morphisms $\iota_\alpha : X_\alpha \to X$ for each α s.t. for any $Y \in Ob(\mathcal{C})$ and any morphisms $\varphi_\alpha : X_\alpha \to Y$ there is unique morphisms $\varphi : X \to Y$ s.t. for each α we have $\varphi\iota_\alpha = \varphi_\alpha$, that is, the following diagram is commutative:



Unlike products, coproducts in familiar categories have rather different descriptions.

Examples: (1) Let $\mathcal{C}$ be the category of sets. Then $\sqcup_{\mathcal{C}} X_\alpha$ is the disjoint union of $\{X_\alpha\}$ (as the notation suggests).

(2) Let $\mathcal{C}$ be the category of groups. Then $\sqcup_{\mathcal{C}} X_\alpha = \star X_\alpha$, the free product of $\{X_\alpha\}$. Informally, this means that given $\alpha' \neq \alpha$, there are no relations between the images of X_α and $X_{\alpha'}$ inside $\star X_\alpha$.

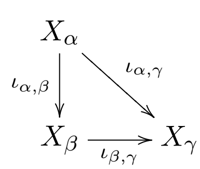
(3) Let $\mathcal{C}$ be the category of abelian groups. Then $\sqcup_{\mathcal{C}} X_\alpha = \oplus X_\alpha$, the direct sum of $\{X_\alpha\}$

(4) Let R be a commutative ring with 1, and let $\mathcal{C} = R - \text{COMMALG}$ be the category of commutative R-algebras. Then $\sqcup_{\mathcal{C}} X_\alpha = \otimes X_\alpha$, the tensor product of $\{X_\alpha\}$.

25.3. **Motivating direct limits.** Let Y be a set and let $\{X_\alpha\}_{\alpha\in A}$ be a collection of subsets of Y which form a chain, that is, for any α, β we have $X_\alpha \subseteq X_\beta$ or $X_\beta \subseteq X_\alpha$. Then we can consider $X = \cup X_\alpha$, the union of X_α as subsets of Y. Our goal is to find a characterization of X similar to that of the disjoint union $\sqcup X_\alpha$.

Let $\leq$ be the order relation on the index set A defined by $\alpha \leq \beta$ if and only if $X_\alpha \subseteq X_\beta$. Note that $\leq$ is a total order on A since $\{X_\alpha\}$ is a chain.

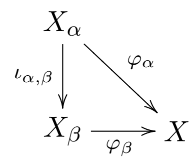
For each $\alpha, \beta \in A$ with $\alpha \leq \beta$ let $\iota_{\alpha,\beta} : X_\alpha \to X_\beta$ be the inclusion map. Note that for any $\alpha \leq \beta \leq \gamma$ the following diagram is commutative:



Now suppose we are given another set Y and maps $\varphi_\alpha : X_\alpha \to Y$ for each $\alpha \in A$. The natural question is

$$\text{when does there exist a map } \varphi : X = \cup X_\alpha \to Y \text{ s.t. } \varphi_{|X_\alpha} = \varphi_\alpha \text{ for } \alpha \in A?$$

Clearly, such φ exists if and only if $(\varphi_\beta)_{|X_\alpha} = \varphi_\alpha$ for any $\alpha \leq \beta$. Equivalently, φ exists if and only if for any $\alpha \leq \beta$ the following diagram is commutative:



Thus the union $X = \cup X_\alpha$ satisfies certain universal property similar to the one in the definition of coproduct, except that instead of considering arbitrary collections of morphisms $\varphi_\alpha : X_\alpha \to Y$ (where Y is another set), one only considers the collections satisfying the compatibility condition (25.3). This analysis provides a motivation for the concept of direct limit, which will be given in the next lecture.