2. Basic properties of modules

2.1. **Isomorphism theorems for modules.** All four isomorphism theorems for groups have direct counterparts for modules. In fact, their statements are even easier since normality assumption is already "included" in the definition of a submodule.

One can obtain the statements of the isomorphism theorems for modules as follows – take the statements of the isomorphism theorems for groups, specialize to the case of *abelian* groups, using additive notation and removing vacuous assumptions and replace every occurrence of group (resp. subgroup) by R-module (resp. R-submodule).

For instance, here are the first and fourth isomorphism theorems for modules.

Theorem (First isomorphism theorem for modules). *If* $\varphi : M \to N$ *is a homomorphism of* R*-modules, then*

- $\operatorname{Ker}\varphi$ *is a submodule of* M*,*
- $\operatorname{Im}\varphi = \varphi(M)$ *is a submodule of* N*,*
- $M/Ker\varphi \cong \varphi(M)$*.*

Theorem (Fourth isomorphism theorem for modules). *Let* M *be a* R*-module and* N *a submodule of* M*. Then there is an inclusion preserving bijection*

$$\text{submodules of } M \text{ containing } N \longleftrightarrow \text{submodules of } M/N.$$

2.2. **Generating and linearly independent subsets of modules.**

Definition. Let M be an R-module and X a subset of M. The the smallest submodule of R containing X will be called the submodule of M generated by X and denoted by RX. Clearly,

$$RX = \{r_1x_1 + \ldots r_kx_k : r_i \in R, x_i \in X\}.$$

Definition. Let M be an R-module and X a subset of M. We will say that

(i) X generates M (as an R-module) if $RX = M$
(ii) X is linearly independent (over R) if equality $r_1x_1 + \ldots r_kx_k = 0$ where $r_i \in R$ and $x_1, \ldots, x_k$ are *distinct* elements of X implies that $r_i = 0$ for all i
(iii) X is a basis of M if X is linearly independent and generates M.

Proposition 2.1. *Assume that R is a field and M an R-module (= R-vector space). Then R has a basis. Furthermore,*

(a) *Any generating subset of M contains a basis of M*
(b) *Any linearly independent subset of M can be extended to a basis of M.*

Proposition 2.1 in the case of finite dimensional vector spaces is a standard fact from undergraduate linear algebra. For general vector spaces it is a consequence of the following result.

Claim. *Let R be a field, M and R-vector space, X a generating set of M, and Y a linearly independent subset of X. Then M has a basis B with $Y \subseteq B \subseteq X$.*

Proof. Apply Zorn's lemma to the set of linearly independent sets Z, with $Y \subseteq Z \subseteq X$, ordered by inclusion. □

If R is not a field, Proposition 2.1 becomes completely false. Here are some examples.

1. Let $R = \mathbb{Z}$ and $n \geq 2$ an integer. Then $\mathbb{Z}/n\mathbb{Z}$ is a $\mathbb{Z}$-module with no non-empty linearly independent subsets.

2. Let $R = \mathbb{Z}$ and $M = R$. Then M has a basis X, in fact there are just two possibilities: $X = \{1\}$ or $X = \{-1\}$. However,
   $\{2, 3\}$ is a generating set which does not contain a basis.
   $\{2\}$ is a linearly independent subset which does not extend to a basis

2.3. **Free modules.**

Definition. An R-module is called <u>free</u> if it has a basis.

By Proposition 2.1 if R is a field, then every R-module is free. We shall now describe the structure of free R-modules over arbitrary rings.

Definition (Direct sum of modules)**.**
1. Finite case: If $M_1, \ldots, M_k$ are R-modules, their <u>(external) direct sum</u>

$$M_1 \oplus \ldots \oplus M_k = \{(m_1, \ldots, m_k) : m_i \in M_i\}$$

is an R-module with componentwise addition and R-action given by

$$r(m_1, \ldots, m_k) = (rm_1, \ldots, rm_k).$$

2. General case: if X is any set and $\{M_x\}_{x \in X}$ is a collection of R-modules, define $\bigoplus_{x \in X} M_x$ to be the set of functions

$\{f : X \to \cup M_x \; : \; f(x) \in M_x$ for each x and the set $\{x : f(x) \neq 0\}$ is finite$\}$.

This is an R-module with pointwise addition and R-action given by

$$(rf)(x) = rf(x)$$

Theorem 2.2. *Let M be an R-module and ω a cardinal number. The following are equivalent.*

(i) *M has a basis of cardinality ω*
(ii) *M is isomorphic to the direct sum of ω copies of R.*

Proof. "(i)$\Rightarrow$ (ii)" Suppose M has a basis X with $|X| = \omega$. Let $M' = \oplus_{x\in X} R_x$ where each R_x is a copy of R.
Take $m \in M$. Since X is a basis, we can uniquely write $m = \sum_{x\in X} r_x x$ where each $r_x \in R$ and only finitely many r_x are 0. Let $\varphi_m \in M'$ be the function defined by $\varphi_m(x) = r_x$ for each x.
We get a map $\varphi : M \to M'$ given by $m \mapsto \varphi_m$. It is routine to check that φ is an isomorphism of R-modules.

"(ii)$\Rightarrow$ (i)" Let X be any set with $|X| = \omega$, and let $M' = \oplus_{x\in X} R_x$ where each R_x is a copy of R. Thus by assumption $M \cong M'$.
For each $x \in X$ let $e_x \in M'$ be the function defined by

$$e_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

It is then easy to check that $\{e_x\}_{x\in X}$ is a basis of M'. Thus M' has a basis of cardinality ω, and the same is true for any module isomorphic to M. □

2.4. **Categorical characterization of free modules.**

Definition. Let M be an R-module and X a subset of M. We will say that M is <u>categorically free on X</u> if for any R-module N, any map $f : X \to N$ uniquely extends to an R-module homomorphism $f_* : M \to N$.

Lemma 2.3 (Uniqueness of categorically free modules)**.** *Suppose that M is categorically free on X and M' is categorically free on X'. Assume that $|X| = |X'|$. Then $M \cong M'$; moreover, there is an R-module isomorphism from $\varphi : M \to M'$ such that $\varphi(X) = X'$.*

Proof. This is a standard diagram argument (similar to what we did for free groups). □

Theorem 2.4. *Let M be an R-module and X a subset of M. The following are equivalent:*

(i) *X is a basis of M.*
(ii) *Any element $m \in M$ can be uniquely written as $m = \sum_{x\in X} r_x x$ where $r_x \in R$ and only finitely many r_x are nonzero.*

(iii) *M is categorically free on X.*

Proof. "(i)$\Rightarrow$ (ii)" is easy (in fact, we already used it).
"(ii)$\Rightarrow$ (iii)" is also easy – just set $f_*(\sum_{x\in X} r_x x) = \sum_{x\in X} r_x f(x)$.
"(iii)$\Rightarrow$ (i)" Let M' be the direct sum of $|X|$ copies of R. By Theorem 2.2 M' has a basis X' with $|X'| = |X|$. Hence by implication "(i)$\Rightarrow$ (iii)" (which we already proved) M' is categorically free on X'.
By Lemma 2.3 there is an isomorphism $\varphi : M' \to M$ such that $\varphi(X') = X$. Clearly, an R-module isomorphism sends a basis to a basis, and thus X must be a basis of M. □

Combining Theorems 2.2 and 2.4, we obtain three equivalent characterizations of free modules

Corollary 2.5. *Let M be an R-module. The following are equivalent:*

(i) *M is free, that is, M has a basis*
(ii) *M is categorically free (on some X)*
(iii) *M is isomorphic to the direct sum of several copies of R.*