## 15. Algebraic closures

We begin with one more result on algebraic extensions.

**Lemma 15.1.** *Suppose that $K/F$ and $L/K$ are algebraic extensions. Then the extension $L/F$ is also algebraic.*

*Proof.* In the case of finitely generated extensions this is immediate by Lemma 14.5. In general we argue as follows.

Take any $\beta \in L$. We are given that $\beta$ is algebraic over $K$, so there exists $p(x) = p_n x^n + \ldots + p_0 \in K[x]$ such that $p(\beta) = 0$. Let $K' = F(p_0, \ldots, p_n) \subseteq K$. Then $\beta$ is also algebraic over $K'$, so the extension $K'(\beta)/K$ is finite. The extension $K'/F$ is algebraic and finitely generated, hence finite by Lemma 14.5. Thus, by Lemma 14.1, $K'(\beta)/F$ is finite, and so $\beta$ is algebraic over $F$ again by Lemma 14.5. □

### 15.1. **Relative algebraic closures.**

**Definition.** Let $K/F$ be a field extension. The set $\overline{F}_K = \{\alpha \in K : \alpha \text{ is algebraic over } F\}$ is called the relative algebraic closure of $F$ in $K$.

**Corollary 15.2.** *Let $K/F$ be a field extension and let $\alpha, \beta \in K$, $\beta \neq 0$, be algebraic over $F$. Then $\alpha \pm \beta$, $\alpha\beta$ and $\frac{\alpha}{\beta}$ are also algebraic over $F$. In other words, the relative algebraic closure $\overline{F}_K$ is a subfield of $K$.*

*Proof.* The elements $\alpha \pm \beta$, $\alpha\beta$ and $\frac{\alpha}{\beta}$ all lie in the field $F(\alpha, \beta)$. Since $\alpha$ and $\beta$ are algebraic over $F$, the extension $F(\alpha, \beta)/F$ is algebraic by Lemma 14.5. □

Example 15.1 The element $\alpha = \sqrt{2} + \sqrt{3}$ is algebraic over $\mathbb{Q}$. Direct verification is not so obvious: $\alpha^2 = 5 + 2\sqrt{6}$, whence $(\alpha^2 - 5)^2 - 24 = 0$.

### 15.2. **Embedding into algebraically closed fields.**

**Lemma 15.3.** *Let $F$ be a field and $p_1, \ldots, p_n \in F[x]$ a finite collection of non-constant polynomials. Then there exists a finite extension $K/F$ such that each $p_i$ has a root in $K$.*

*Proof.* *Case 1:* $n = 1$. WOLOG we can assume that each $p_1$ is irreducible. Let $K_1 = F[x]/(p_1(x))$. Then $K_1$ is a field, and we have a natural embedding $F \to K_1$ via $F \to F[x] \to F[x]/(p_1(x))$. If $\alpha = \bar{x}$ is the image of $x$ in $K_1$, then $p_1(\alpha) = p_1(\bar{x}) = \overline{p_1(x)} = 0$. Thus, $p_1$ has a root in $K_1$. Note that $[K_1 : F] = \deg p_1(x)$ is finite.

*General case.* By consecutive applications of Case 1 we can construct an ascending chain of fields $F = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_n$ such that $p_i$ has a root in $K_i$. Hence each of the polynomials $p_1, \ldots, p_n$ has a root in $K_n$. By construction each extension $K_i/K_{i-1}$ is finite, so $K_n/F$ is also finite. Thus, $K = K_n$ has the desired property. □

**Theorem 15.4.** *Any field $F$ can be embedded in an algebrically closed field.*

*Proof. Step 1:* Construct an extension $K_1/F$ s.t. every non-constant polynomial from $F[x]$ has a root in $K_1$.

Let $S$ be the set of all non-constant polynomials in $F[x]$. For each $f \in S$ introduce a formal variable $x_f$ and let $R = F[\{x_f\}_{f \in S}]$ be the ring of polynomials over $F$ in $\{x_f\}$.

Let $I$ be the ideal of $R$ generated by $\{f(x_f) : f \in S\}$. We claim that $I \neq R$. Suppose not. Then there exist $f_1, \ldots, f_n \in S$ and $g_1, \ldots, g_n \in R$ s.t.

$$g_1 f_1(x_{f_1}) + \ldots + g_n f_n(x_{f_n}) = 1. \qquad (*)$$

By Lemma 15.3 there exists an extension $K/F$ in which $f_1$ has some root $\alpha_1$, …, $f_n$ has some root $\alpha_n$. Now take (*), set $x_{f_i} = \alpha_i$ for $1 \leq i \leq n$, and set $x_f$ to be any element of $F$ for $f \notin \{f_i\}$. We get $0 = 1$, which is impossible.

Once we proved that $I \neq R$, we know that there is a maximal ideal $M$ s.t. $I \subseteq M$. Let $K_1 = R/M$. Then $K_1$ is a field containing an (isomoprhic copy of) $F$ and each $f \in S$ has a root in $K_1$. Indeed, if $\overline{x_f}$ is the image of $x_f$ in $K_1$, then $f(\overline{x_f}) = \overline{f(x_f)} = 0$ as $f(x_f) \in I \subseteq M$. This completes Step 1.

*Step 2:* Apply Step 1 to $K_1$, that is, construct an extension $K_2/K_1$ s.t. every polynomial in $K_1[x]$ has a root in $K_2$. Then apply Step 1 to $K_2$ etc.

$$F \subseteq K_1 \subseteq K_2 \subseteq \ldots$$

Let $K = \bigcup_{n=1}^{\infty} K_n$. Then $K$ is a field being the union of an ascending chain of fields. Given a non-constant polynomial $p \in K[x]$, there exists $n \in \mathbb{N}$ s.t. $p \in K_n[x]$. By construction $p$ has a root in $K_{n+1}$ and hence has a root in $K$. Thus $K$ is algebraically closed. □

### 15.3. **Existence of algebraic closures.**

**Definition.** Let $F$ be a field. An extension field $\overline{F}$ of $F$ is called an <u>algebraic closure of $F$</u> if

(i) $\overline{F}$ is algebraically closed
(ii) the extension $\overline{F}/F$ is algebraic

**Theorem 15.5.** *Any field $F$ has an algebraic closure.*

*Proof.* By Theorem 15.4 $F$ can be embedded in an algebraically closed field $K$. Let $L = \{\alpha \in K : \alpha \text{ is algebraic over } F\}$ be the relative algebraic closure of $F$ in $K$. Then $L$ is a field by Corollary 15.2 and the extension $L/F$ is algebraic. It remains to show that $L$ is algebrically closed.

Take any non-constant polynomial $p(x) \in L[x]$. We know that $p(x)$ has a root $\beta \in K$. We will be done if we show that $\beta \in L$.

Since $\beta$ is algebraic over $L$, the extension $L(\beta)/L$ is algebraic. Since $L/F$ is also algebraic, by Lemma 15.1 $L(\beta)/F$ is algebraic. In particular, $\beta$ is algebraic over $F$, so by definition $\beta \in L$. □

Uniqueness of algebraic closures will be proved in the next lecture.