12. Jordan canonical form of matrices and linear transformations.

Let F be a field, V a f.d. vector space over F and $T \in \mathfrak{gl}(V)$ an F-linear transformation from V to V. As before let V_T denote V considered as $F[x]$-module where x acts as T.

Recall that the existence of the rational canonical form (RCF) of T was derived from the invariant factors decomposition of the $F[x]$-module V_T. Today we shall use the decomposition of V_T in the elementary divisors form to establish the existence of the *Jordan canonical form* (JCF). While RCF exists over any field, to ensure the existence of JCF we need to assume that F is algebraically closed. This is not a very serious restriction since any field can be embedded into an algebraically closed field – we will show this later in the course.

Definition. A field F is called algebraically closed if any non-constant polynomical in $F[x]$ has a root in F. Equivalently, F is algebraically closed if any irreducible polynomial in $F[x]$ has degree 1.

12.1. **Existence and uniqueness of Jordan canonical form.** So, let F be an algebraically closed field, V a f.d. F-vector space and $T \in \mathfrak{gl}(V)$. We apply the classification of modules over PID in ED form to the module V_T. Since all irreducible polynomials in $F[x]$ are linear, we get that there exist $\lambda_1, \ldots, \lambda_k \in F$ (not necessarily distinct) and positive integers $d_1, \ldots, d_k$ such that

$$V_T = V_1 \oplus \ldots \oplus V_k \text{ where } V_i \cong F[x]/(x-\lambda_i)^{d_i} \text{ as } F[x]\text{-modules.}$$

As in RCF case, each V_i is T-invariant. If we let $T_i = T_{|V_i} \in \mathfrak{gl}(V_i)$, choose a basis Ω_i of V_i for each i and let $\Omega = \Omega_1 \sqcup \ldots \sqcup \Omega_k$, then Ω is a basis of V and

$$[T]_\Omega = \begin{pmatrix} [T_1]_{\Omega_1} & 0 & \ldots & 0 \\ 0 & [T_2]_{\Omega_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & [T_k]_{\Omega_k} \end{pmatrix}$$

Thus, as with RCF we are reduced to the case when

$$V_T \cong F[x]/(x-\lambda)^d \text{ as } F[x]\text{-modules.}$$

For each $0 \le i \le d-1$ let $e_i = \overline{(x-\lambda)^i}$ where $\overline{p(x)}$ is the image of $p(x)$ in V_T. Let $\Omega = \{e_{d-1}, \ldots, e_0\}$ (in this order!) Then Ω is an F-basis of V, and the action of T on Ω is given by

$T(\overline{e_i}) = \overline{x(x-\lambda)^i} = \overline{(x-\lambda)^{i+1} + \lambda(x-\lambda)^i} = \lambda e_i + e_{i+1}$ if $i < d-1$ and $T(\overline{e_{d-1}}) = \lambda e_{d-1}$.

So,

$$[T]_\Omega = \begin{pmatrix} \lambda & 1 & \dots & 0 & 0 \\ 0 & \lambda & 1 & & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \lambda & 1 \\ 0 & \dots & \dots & 0 & \lambda \end{pmatrix}$$

This $d \times d$ matrix will be denoted by $J(d, \lambda)$. Matrices of this form are called Jordan blocks.

Definition. A matrix $A \in Mat_n(F)$ is said to be in Jordan canonical form if A is block-diagonal where each block is a Jordan block.

Theorem 12.1 (Existence and uniqueness of JCF)**.** *Let V be a f.d. vector space over an algebraically closed field and $T \in \mathfrak{gl}(V)$. Then there is a basis Ω of V and a matrix J in Jordan canonical form s.t. $[T]_\Omega = J$. The matrix J is called the Jordan canonical form of T – it is unique up to permutation of Jordan blocks.*

Proof. The existence part is proved above. The uniqueness part can be deduced from the uniqueness part in the PID classificaiton theorem (ED form), but we shall also give a different argument later on. □

Here is the matrix form of Theorem 12.1:

Theorem 12.1'. *Assume that F is algebraically closed. Then any matrix $A \in Mat_n(F)$ is similar to a matrix in JCF, which is unique up to permutation of Jordan blocks.*

From now on we will primarily state our results for matrices (instead of linear transformations).

12.2. **Relation of JCF to minimal and characteristic polynomial.**

Lemma 12.2. *Let F be an algebraically closed field and $A \in Mat_n(F)$. Let $Spec(A)$ be the set of all eigenvalues of A.*

(a) *Let $\lambda \in F$. Then $\lambda \in Spec(A)$ if and only if $JCF(A)$ contains at least one Jordan λ-block (that is, a block of the form $J(d, \lambda)$).*
(b) *The characteristic polynomial $\chi_A(x)$ is equal to $\prod_{\lambda \in Spec(A)} (x-\lambda)^{s_\lambda}$ where s_λ is the sum of sizes of all Jordan λ-blocks in $JCF(A)$;*

(c) *The minimal polynomial* $\mu_A(x)$ *is equal to* $\prod_{\lambda \in Spec(A)} (x-\lambda)^{m_\lambda}$ *where* m_λ *is the maximum size of a Jordan* λ*-block in* $JCF(A)$.

Proof. Since $Spec(A)$, $\chi_A(x)$ and $\mu_A(x)$ do not change under conjugation, we can assume that A is in JCF. In this case (a) and (b) are obvious.

It is also clear that $\mu_A(x)$ is the least common multiple of the minimal polynomials of the Jordan blocks. Thus, to prove (c) it is enough to show that if $A = J(d, \lambda)$ for some $d \in \mathbb{N}$ and $\lambda \in F$, then $\mu_A(x) = (x - \lambda)^d$, and the latter can be verified by a simple computation. □

12.3. **Computing JCF using ranks.** Let F be an algebraically closed field and $A \in Mat_n(F)$. We can determine the eigenvalues of A by computing the characterisitc polynomial $\chi_A(x)$. The sizes of the Jordan blocks are often easy to compute via the following observation.

Lemma 12.3. *For each* $\lambda \in F$ *and* $k \in \mathbb{N}$ *let* $n_A(k, \lambda)$ *be the number of Jordan* λ*-blocks of size* $\geq k$ *in* $JCF(A)$. *Then*

$$n_A(k, \lambda) = \mathrm{rk}((A - \lambda I)^{k-1}) - \mathrm{rk}((A - \lambda I)^k) \qquad (***)$$

(here I *is the identity matrix of suitable size)*

Remark: Formula (***) holds for all $\lambda \in F$, not only for $\lambda \in Spec(A)$.

Proof. As before, we can assume that A is in JCF.
Case 1: A has just one Jordan block, that is, $A = J(d, \mu)$.

If $\mu \neq \lambda$, then clearly $n_A(k, \lambda) = 0$ for each k, and on the other hand the matrix $(A - \lambda I)^k$ has maximal rank being invertible. Thus, (***) holds.

If $\lambda = \mu$, then $n_A(k, \lambda) = 1$ for $k \leq d$ and $n_A(k, \lambda) = 0$ for $k > d$. On the other hand, direct computation shows that

$$\mathrm{rk}((A - \lambda I)^k = \begin{cases} d - k & \text{if } k \leq d \\ 0 & \text{if } k > d \end{cases}$$

Again, we see that (***) holds for each k.
Case 2: A has more than one Jordan block.

We can put A into a block-diagonal form with non-trivial blocks B and C. Note that

(i) $n_A(k, \lambda) = n_B(k, \lambda) + n_C(k, \lambda)$
(ii) $\mathrm{rk}((A - \lambda I)^k) = \mathrm{rk}((B - \lambda I)^k) + \mathrm{rk}((C - \lambda I)^k)$

By induction (***) holds for both B and C. Combining this fact with (i) and (ii), we get that (***) also holds for A. □

Remark: Lemma 12.3 gives an alternative proof of the uniqueness of JCF (up to permutation of Jordan blocks).

Corollary 12.4. *Any matrix $A \in Mat_n(F)$ is similar to its transpose A^T.*

Proof. Let F' be an algebraically closed field containing F. By Corollary 10.3 it is enough to prove that A and A^T are similar in $Mat_n(F')$.
Note that for each $\lambda \in F'$ and $k \in \mathbb{N}$ we have $((A-\lambda I)^k)^T = (A^T - \lambda I)^k$ (since $(BC) = C^T B^T$), and therefore

$$\mathrm{rk}((A^T - \lambda I)^k) = \mathrm{rk}((A - \lambda I)^k).$$

Lemma 12.3 now implies that A and A^T have the same JCF (over F') and hence must be similar in $Mat_n(F')$. □

12.4. **Root subspaces.** In the last two subsections we fix an algebraically closed field F, a f.d. vector space V over F and $T \in \mathfrak{gl}(V)$. For each $\lambda \in F$ let

$$V_\lambda = \{v \in V : (T - \lambda I)^k v = 0 \text{ for some } k \in \mathbb{N}\}.$$

The subspaces V_λ are called root subspaces (or generalized eigenspaces) of T.
Note that $V_\lambda \neq \{0\}$ if and only if λ is an eigenvalue of T and that V_λ always contains the eigenspace $E_\lambda = \{v \in V : Tv = \lambda v\}$.

Observation 12.5. *The following are equivalent:*

- (i) *T is diagonalizable, that is, T is represented by a diagonal matrix with respect to some basis*
- (ii) *All Jordan blocks in $JCF(T)$ have size* 1
- (iii) *$V_\lambda = E_\lambda$ for each eigenvalue λ of T*
- (iv) *The minimal polynomial $\mu_T(x)$ has no multiple roots.*

Proof. Follows immediately from what we have already proved. □

Lemma 12.6. *The space V is a direct sum of the root subspaces V_λ:*

$$V = \oplus_{\lambda \in Spec(T)} V_\lambda.$$

Proof. By Theorem 12.1 we have a decomposition

$$V = V_1 \oplus \ldots \oplus V_k \qquad (***)$$

such that each V_i is T-invaraint, and if $T_i = T_{|V_i}$, then there is a basis Ω_i of V_i such that $[T_i]_{\Omega_i}$ is some Jordan block $J(d_i, \lambda_i)$.

It is clear that for each $\lambda \in Spec(T)$ the root subspace V_λ is the (direct) sum of all V_i for which $\lambda_i = \lambda$. This combined with (***) yields the lemma. □

12.5. **A few words on computing a Jordan basis.**

Definition. A basis Ω of V is called a Jordan basis for T if $[T]_\Omega$ is in JCF.

Below we discuss how to compute Jordan basis in two simple cases. A few more complicated cases will be discussed in the homework, and a general algorithm is given in Kevin McCrimmon's 'General exam guide'.

Case 1: T is diagonalizable. In this case we simply compute eigenvalues of T, then for each eigenvalue λ compute the eigenspace E_λ (by solving the equation $Tv = \lambda v$) and pick a basis for each E_λ. By Observation 12.5 and Lemma 12.6 the union of these bases is a Jordan basis for T.

Case 2: $JCF(T)$ has just one block. Let λ be the unique eigenvalue of T. Clearly, we may replace T by $T - \lambda I$ (since any Jordan basis for $T - \lambda I$ is also a Jordan basis for T), and thus we may assume that $\lambda = 0$.

Let $n = \dim V$. We know that if $\{e_0, \ldots, e_{n-1}\}$ is a Jordan basis for T, then $Te_0 = 0$ and $Te_i = e_{i-1}$ for $i > 0$. The following lemma is a partial converse of this statement which also provides an algorithm for finding a Jordan basis:

Lemma 12.7. *The following hold:*

(i) *For any* $0 \leq k \leq n$ *we have* $\operatorname{Im} T^k = \operatorname{Ker} T^{n-k}$

(ii) *Let* v_{n-1} *be any vector which does not lie in* $\operatorname{Im} T = \operatorname{Ker} T^{n-1}$*, and set* $v_i = Tv_{i+1}$ *for* $n-2 \geq i \geq 0$*. Then* $\{v_0, \ldots, v_{n-1}\}$ *is a Jordan basis for* T*.*

Proof. This is part of Homework #6. □