Homework Assignment # 9.

Plan for the next week:

Lecture 22 (April 13): Finite fields (the end of 13.5 and 14.3)

Lecture 23 (April 15): Cyclic Galois extensions (Lang, Section 6.6)

Problems, to be submitted by Thu, April 15th.

Problem 1: Let F be a field, let $f(x) \in F[x]$ be an irreducible (over F) separable polynomial of degree n, and let K be a splitting field of $f(x)$.

(a) Prove that $|\mathrm{Gal}\,(K/F)|$ is a multiple of n and divides $n!$

(b) Let $n = 3$. Prove that $\mathrm{Gal}\,(K/F)$ is isomorphic to either $\mathbb{Z}/3\mathbb{Z}$ or S_3.

(c) Let $n = 4$ and assume that $|\mathrm{Gal}\,(K/F)| = 8$. Determine the isomorphism class of $\mathrm{Gal}\,(K/F)$.

Problem 2: Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree n, and let K be a splitting field of $f(x)$. Label the roots of $f(x)$ by $1, \ldots, n$ (in some order), and let $\iota : \mathrm{Gal}\,(K/\mathbb{Q}) \to S_n$ be the associated embedding.

(a) Assume $f(x)$ has at least one non-real root. Prove that the complex conjugation is an element of $\mathrm{Gal}\,(K/\mathbb{Q})$ of order 2.

(b) Assume that $f(x)$ has precisely two non-real roots. Prove that the image of the complex conjugation under the embedding ι is a transposition.

(c) Suppose that $n = \deg(f)$ is prime and again assume that $f(x)$ has precisely two non-real roots. Prove that $\mathrm{Gal}\,(K/\mathbb{Q})$ is isomorphic to S_n. **Hint:** $\mathrm{Gal}\,(K/\mathbb{Q})$ must contain an element of order n (why?)

Problem 3: Let K be the splitting field of $f(x) = x^4 - 2$ over $\mathbb{Q}$. Recall that the Galois group $\mathrm{Gal}\,(K/\mathbb{Q})$ has already been described elementwise in Problem#2 of Homework#8 (you may refer to that problem freely).

(a) Choose an order on the set of roots of $x^4 - 2$ and describe the associated embedding of $\mathrm{Gal}\,(K/\mathbb{Q})$ to S_4.

(b) Describe all subgroups of $\mathrm{Gal}\,(K/\mathbb{Q})$ and the corresponding subfields of K.

Problem 4: Let K/F and L/F be Galois extensions.

(a) Prove that the extension KL/F is also Galois and there is a natural embedding $\iota : \mathrm{Gal}\,(KL/F) \to \mathrm{Gal}\,(K/F) \times \mathrm{Gal}\,(L/F)$.

(b) Assume now that K/F and L/F are both finite. Prove that the map ι in (a) is an isomorphism if and only if $K \cap L = F$.

Problem 5 (practice): DF, Problem 6 on page 582.

Problem 6: DF, Problem 17 on pages 582-583.

Problem 7 (bonus): Let $K = \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_t})$ where $\{p_1, \ldots, p_t\}$ are distinct primes. Prove that $[K : \mathbb{Q}] = 2^t$ using Galois theory and determine $\mathrm{Gal}\,(K/\mathbb{Q})$. **Hint:** Use something we proved about finite abelian p-groups in class last semester.