Homework Assignment # 8.

Plan for next week: more on separable extensions, start Galois theory.

Note: this assignments is based on parts of 13.5 and 13.6 in DF that we have not discussed in class so far.

Problems, to be submitted by Thu, April 2nd.

Problem 1:

(a) Let F be a field, let $\overline{F}$ be an algebraic closure of F and $\sigma : \overline{F} \to \overline{F}$ an F-embedding. Prove that $\sigma(\overline{F}) = \overline{F}$, and thus σ is an automorphism of $\overline{F}$.

(b) Prove that for any field F any two algebraic closures of F are F-isomorphic. **Hint:** Use (a) and the Main Extension Lemma.

Problem 2: For each of the following polynomials $f(x) \in \mathbb{Q}[x]$ let $K \subseteq \mathbb{C}$ be the splitting field of f over $\mathbb{Q}$:

(i) $f(x) = x^n - 1$, $n \geq 2$

(ii) $f(x) = x^4 + 3x^3 + 4x^2 + 3x + 3$

(iii) $f(x) = x^4 - 2$

Find the degree $[K : \mathbb{Q}]$ and express K in the form $\mathbb{Q}(\alpha)$ or $\mathbb{Q}(\alpha, \beta)$

Problem 3 (practice): (a) Prove that if $K/\mathbb{Q}$ is any field extension, then any automorphism of K must fix $\mathbb{Q}$ elementwise.

(b) Prove that the automorphism group $\mathrm{Aut}(\mathbb{R}/\mathbb{Q})$ is trivial. See [DF, Problem 7, p.567] for a sketch of the proof.

Problem 4: Let K/F be an algebraic extension. Prove that K/F is normal if and only if for any algebraic extension L/K and any F-automorphism $\sigma \in \mathrm{Aut}_F(L)$ we have $\sigma(K) = K$.

Problem 5: Let K/F be a field extension, and let K_1 and K_2 be subfields of K containing F such that the extensions K_1/F and K_2/F are normal. Prove that the extensions K_1K_2/F and $K_1 \cap K_2/F$ are also normal.

Problem 6: Problems 10 and 11 in [DF, p.556]

Problem 7: Problem 13 in [DF, p.556].