Homework Assignment # 7.

Plan for next week:

Lecture 16: Uniqueness of algebraic closures, splitting fields (roughly 13.4)
Lecture 17: Normal and separable extensions (partially contained in 13.5)

Problems, to be submitted by Thursday, March 25th.

Remark 0: Recall that if K/F and L/K are finite field extensions, then $[L:F]=[L:K][K:F]$. In particular, both $[K:F]$ and $[L:K]$ divide $[L:F]$. This simple observation turns out to be extremely useful.

Problem 1: (a) Let $K=\mathbb{Q}(\sqrt{2},\sqrt{3})$. Prove that $[K:\mathbb{Q}]=4$.
(b) Let $L=\mathbb{Q}(\sqrt{2}+\sqrt{3})$. Prove that $L=K$ and hence $[L:\mathbb{Q}]=4$ by (a).

Problem 2: Let $S=\{n_1,\ldots,n_k\}$ be a finite set of positive integers ≥ 2 and let $K=\mathbb{Q}(\sqrt{n_1},\ldots,\sqrt{n_k})$.

(a) Prove that $[K:\mathbb{Q}]=2^m$ for some $m\leq k$ and the set $P(S)=\{1\}\cup\{\sqrt{n}: n \text{ is a product of distinct elements of } S\}$ spans K over $\mathbb{Q}$.

(b) For each $0\leq j\leq k$ let $\mathbb{Q}_j=\mathbb{Q}(\sqrt{n_1},\ldots,\sqrt{n_j})$ (we set $\mathbb{Q}_0=\mathbb{Q}$). Prove that $[K:\mathbb{Q}]<2^k$ if and only if n_1 is a complete square or there exists $2\leq i\leq k$ s.t. $\sqrt{n_i}=a+b\sqrt{n_{i-1}}$ for some $a,b\in\mathbb{Q}_{i-2}$.

(c) Assume that the elements of S are pairwise relatively prime. Prove that $[K:\mathbb{Q}]=2^k$. **Hint:** Use (b) and induction on $k=|S|$.

Problem 3: Let F be a field, and let α be an algebraic element of odd degree over F (where the degree of α over F is $[F(\alpha):F]$). Prove that $F(\alpha^2)=F(\alpha)$.

Problem 4: Let K/F be an algebraic extension.

(a) Prove that if $F\subseteq R\subseteq K$ and R is a subring, then R must be a subfield. **Hint:** This follows very easily from what we proved in class.

(b) Show that (a) would be false without the assumption K/F is algebraic.

Problem 5: Let K/F be a finite field extension, $n=[K:F]$, and fix some basis $\Omega=\{\alpha_1,\ldots,\alpha_n\}$ for K over F. For any $\alpha\in K$ define $T_\alpha:K\to K$ by $T_\alpha(\beta)=\alpha\beta$. Note that $T_\alpha\in End_F(K)$. Let $A_\alpha=[T_\alpha]_\Omega\in Mat_n(F)$ be the matrix of T_α with respect to Ω.

(a) (practice) Prove that the map $K \to Mat_n(F)$ given by $\alpha \mapsto A_\alpha$ is an injective ring homomorphism.

(b) Prove that the minimal polynomial of α over F and the minimal polynomial of A_α coincide.

(c) Find the minimal polynomial of the matrix

$$A = \begin{pmatrix} 0 & 3 & 5 & 0 \\ 1 & 0 & 0 & 5 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

without doing extensive computations. **Hint:** Find an extension $K/\mathbb{Q}$ of degree 4, a basis for K over $\mathbb{Q}$ and an element $\alpha \in K$ such that $A_\alpha = A$ in the notations of part (a).

Problem 6: Before doing this problem read about composite fields on pp. 528-529. Let E/F be a field extension, let K_1 and K_2 be subfields of E containing F, and assume that the extensions K_1/F and K_2/F are finite. Let K_1K_2 be the composite field of K_1 and K_2. Prove that the F-algebra $K_1 \otimes_F K_2$ is a field if and only if $[K_1K_2 : F] = [K_1 : F][K_2 : F]$. **Hint:** First show that there exists an F-linear map $\Phi : K_1 \otimes_F K_2 \to K_1K_2$ such that $\Phi(a \otimes b) = ab$ for any $a \in K_1$ and $b \in K_2$. Then explain why Φ is always surjective.