Homework Assignment #6.

Plan for next week: Basic field theory and field extensions (13.1,13.2), start algebraic closures (13.4).

Problems, to be submitted by Thu, March 18th.

Problem 1: V be a finite-dimensional vector space over an algebraically closed field and $T \in \mathfrak{gl}(V)$. Assume that T has just one eigenvalue λ and just one Jordan block. Let $S = T - \lambda I$ and $n = \dim V$.

(a) Prove that $rk(S^k) = n - k$ for $0 \leq k \leq n$. Deduce that $\Im(S^k) = \mathrm{Ker}\,(S^{n-k})$ for $0 \leq k \leq n$.

(b) Let $v \in V$ be any vector which lies outside of $\Im(S) = \mathrm{Ker}\,(S^{n-1})$ (why does such vector exist)? Prove that $\{S^{n-1}v, \ldots, Sv, v\}$ is a Jordan basis for T.

Problem 2: Again let V be a finite-dimensional vector space over an algebraically closed field, $T \in \mathfrak{gl}(V)$ and $n = \dim V$.

(a) Assume that T has unique eigenvalue 0 and two Jordan blocks: a 1×1 block and a 2×2 block (so $n = 3$). Justify the following algorithm for computing a Jordan basis for T: Take any $v \in V \setminus \mathrm{Ker}\,(T)$ and choose $w \in \mathrm{Ker}\,(T)$ such that $\{w, Tv\}$ is a basis for $\mathrm{Ker}\,(T)$ (why is this possible?); then $\{w, Tv, v\}$ is a Jordan basis for T.

(b) Assume that T has unique eigenvalue 0 and two Jordan blocks, both of which are 2×2 (so $n = 4$). State an algorithm for finding a Jordan basis similar to the one in (a).

(c) Assume that for each $\lambda \in Spec(T)$ there is only one Jordan λ-block in $JCF(T)$. Decribe an algoirhtm for computing a Jordan basis of T. **Hint:** You just need a minor generalization of the algorithm in Problem 1.

Problem 3: Compute the Jordan canonical form and a Jordan basis for each of the following matrices over $\mathbb{Q}$:

(a) $\begin{pmatrix} -1 & 3 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$

Problem 4: Let $F = \mathbb{F}_3$ be the field with 3 elements and let A be a 12×12 matrix over F satisfying the following conditions:

$$rk(A) = 10 \qquad rk(A^2) = 9 \qquad rk(A^3) = 9 \tag{1}$$
$$rk(A-1) = 12 \tag{2}$$
$$rk(A-2) = 9 \qquad rk((A-2)^2) = 7 \qquad rk((A-2)^3) = 6 \tag{3}$$

(a) Assume in addition that $\chi_A(x)$ splits completely over F, that is, $\chi_A(x)$ is a product of linear factors in $F[x]$. Find the Jordan canonical form of A.

(b) Find all possible RCFs of matrices A satisfying (1)-(3) (without the extra assumption in (a))

Problem 5: Let V be a finite-dimensional vector space over a field. Let $T : V \to V$ be a diagonalizable linear transformation, let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of T and $V_1, \ldots, V_k$ the corresponding eigenspaces so that $V = V_1 \oplus \ldots \oplus V_k$.

(a) Prove that a subspace U of V is T-invariant $\iff U = \oplus_{i=1}^k (U \cap V_i)$.

(b) Use (a) to show that every T-invariant subspace U of V has a T-invariant complement, that is, there exists a T-invariant subspace W such that $V = U \oplus W$.

Hint for (a): The backwards direction is easy; here is a sketch of the proof in the forward direction. Since $V = V_1 \oplus \ldots \oplus V_k$, each vector $v \in V$ can be uniquely written as $v = \sum_{i=1}^k [v]_i$ where $[v]_i \in V_i$. Express $[Tv]_i$, $[T^2 v]_i$ etc. in terms of $[v]_i$ (easy). Next prove that for any $v \in V$ the vectors $[v]_1, \ldots, [v]_k$ lies in the span of $\{v, Tv, T^2 v, \ldots, T^{k-1} v\}$ – this follows from a basic result on the Vandermonde determinant (look it up on Wikipedia). The last statement easily implies the forward direction in (a).

Problem 6: Let $V = C^\infty(\mathbb{R})$ be the space of all infinitely differentiable functions on the real line and $T = \frac{d}{dx} : V \to V$ the differentiation operator. Find all eigenvalues of T and the corresponding root subspaces (=generalzed eigenspaces). **Hint:** Start by computing the root subspace corresponding to $\lambda = 0$.

Problem 7: (optional) Problem 27 on p.358 of [DF]. It shows that the rank of a free R-module over a non-commutative ring R may not be well defined.