Homework Assignment # 5.

Plan for next week: 12.3 (Jordan canonical form).

Problems, to be submitted by Thu, March 4th.

Problem 1: Let R be a PID. For an R-module M define $\mathrm{rk}(M)$ to be the minimal size of a generating set of M.

(a) Let M be a finitely generated R-module and $R/a_1R\oplus\ldots\oplus R/a_mR\oplus R^s$ its invariant factor decomposition, that is, $a_1,\ldots,a_m$ are nonzero non-units and $a_1\mid a_2\mid\ldots\mid a_m$. Prove that $\mathrm{rk}(M)=m+s$. **Warning:** It is not true in general that $\mathrm{rk}(P\oplus Q)=\mathrm{rk}(P)+\mathrm{rk}(Q)$.

(b) Let F be a free R-module of rank n with basis $e_1,\ldots,e_n$, let N be the submodule of F generated by some elements $v_1,\ldots,v_n\in F$, and let $A\in Mat_n(F)$ be the matrix such that

$$\begin{pmatrix} v_1\\ \vdots\\ v_n\end{pmatrix}=A\begin{pmatrix} e_1\\ \vdots\\ e_n\end{pmatrix}$$

Find a simple condition on the entries of A which holds if and only if $\mathrm{rk}(F/N)=n$.

Problem 2: DF, Problem 6, page 488.

Problem 3: DF, Problem 9, page 489. First two matrices are for practice.

Problem 4: (a) Recall that for a matrix A we denoted by $\chi_A(x)$ and $\mu_A(x)$ its characteristic and minimal polynomials, respectively.

Determine the number of possible RCFs of 8×8 matrices A over $\mathbb{Q}$ with $\chi_A(x)=x^8-x^4$. Explain your argument in detail.

(b) (practice) Determine the number of possible RCFs of 10×10 matrices A over $\mathbb{Q}$ with $\mu_A(x)=x^6-x^2$. Do the same for matrices over $\mathbb{C}$.

Problem 5: (a) (practice) Prove that two 3×3 matrices over some field F are similar if and only if they have the same minimal and characteristic polynomials. Give an example showing that this does not hold for 4×4 matrices.

(b) A matrix A is called idempotent if $A^2=A$. Prove that two idempotent $n\times n$ matrices are similar if and only if they have they same rank. **Hint:** What is the minimal polynomial of an idempotent matrix? How does rank relate to eigenvalue 0?

Problem 6: Find the number of distinct conjugacy classes in the group $GL_3(\mathbb{Z}/2\mathbb{Z})$ and specify one element in each conjugacy class.