Homework Assignment # 4.

Plan for next week: Modules over PID continued (12.1).

Problems, to be submitted by Thu, February 18th.

Problem 1. Let V and W be finite dimensional vector spaces over a field F, let $\{v_1, \dots, v_n\}$ be a basis of V and $\{w_1, \dots, w_m\}$ a basis of W.
Let $\varphi : V \otimes_F W \to Mat_{n\times m}(F)$ be the F-linear transformation such that $\varphi(v_i \otimes w_j) = e_{ij}$ where e_{ij} is the matrix whose (i,j)-entry is equal to 1 and all other entries are equal to 0 (note that such transformation exists and is unique because $\{v_i \otimes w_j : 1 \le i \le n, 1 \le j \le m\}$ is a basis for $V \otimes_F W$; furthermore, φ is an isomorphism since matrices $\{e_{ij}\}$ form a basis of $Mat_{n\times m}(F)$).
Prove that for a matrix $A \in Mat_{n\times m}(F)$ the following are equivalent:

(a) $A = \varphi(v \otimes w)$ for some $v \in V, w \in W$ (note: v and w need not be elements of the above bases)

(b) $rk(A) \le 1$.

Note that this problem yields a one-line solution to Problem 4 from HW#3.

Problem 2. (a) Let R be a commutative ring, let M be an R-module and N its submodule. Prove that M is Noetherian $\iff$ N and M/N are both Noetherian (see 12.1 for the definition of a Noetherian module)
Hint: The forward direction is easy. For the backwards direction, observe that if $\{P_i\}$ is an ascending chain of submodules of M, then $\{P_i \cap N\}$ is an ascending chain of submodules of N and $\{(P_i + N)/N\}$ is an ascending chain of submodules of M/N.
(b) Let R be a commutative Noetherian ring. Use (a) to prove that R^n is a Noetherian module for any $n \in \mathbb{N}$.
(c) Use (a) and (b) to prove Lemma 7.6 from class: if R is Noetherian, then every submodule of a finitely generated R-module is finitely generated.

Problem 3. Let $R = \oplus_{n=0}^{\infty} R_n$ be a graded ring. An element $r \in R$ is called *homogeneous* if $r \in R_n$ for some n.

Any $r \in R$ can be uniquely written as $r = \sum_{n=0}^{\infty} r_n$ where $r_n \in R_n$ and only finitely many r_n's are nonzero. The elements $\{r_n\}$ are called the *homogeneous components of* r.
(a) Let I be an ideal of R. Prove that the following are equivalent:

(i) I is a graded ideal, that is, $I = \oplus_{n=0}^{\infty} I \cap R_n$

(ii) For each $r \in I$ all homogeneous components of r also lie in I

(b) Let I be an ideal of R generated by homogeneous elements (possibly of different degrees). Prove that I is graded.

Problem 4. Prove Proposition 6.5 from the online version of Lecture 6.

Problem 5. Let A be a ring (with 1). A subring B of A is called a *retract* if there exists a surjective ring homomorphism $\varphi : A \to B$ such that $\varphi_{|B} = id_B$, that is, $\varphi(b) = b$ for all $b \in B$.
Now let M and N be two R-modules. Prove that the tensor algebra $T(M)$ is (naturally isomorphic to) a subalgebra of $T(M \oplus N)$ and that this subalgebra is a retract. Also prove the analogous statement about the symmetric algebras.
Problem 6 (practice). Let $F = \mathbb{Z}^3$ and N the $\mathbb{Z}$-submodule of F generated by $(1, 2, 3)$, $(5, 4, 6)$ and $(7, 8, 9)$.
(a) Find compatible bases for F and N, that is, bases satisfying the conclusion of Key Theorem 7.1 from class (= Theorem 4 on page 460 of DF).
(b) Describe the quotient group F/N in the invariant factors form
(c) Describe in the invariant factors form the abelian group given by the presentation $\langle a, b, c \mid a + 2b + 3c = 0, 5a + 4y + 6c = 0, 7a + 8b + 9c = 0\rangle$.