Homework Assignment # 3.

Plan for next week: Tensor, symmetric an exterior algebras (11.5), modules over PID (12.1).

Problems, to be submitted by Thu, February 11th.

Problem 1. Solve Problem 8 on pp.375-376 of [DF]. Note that there is a small mistake (or rather a misprint) in the setup that you need to find and fix.

Problem 2. (a) Let V be a finite-dimensional vector space over $\mathbb{C}$ (complex numbers). Note that V can also be considered as a vector space over $\mathbb{R}$, but $\dim_{\mathbb{R}}(V) = 2\dim_{\mathbb{C}}(V)$. Prove that $V \otimes_{\mathbb{C}} V$ is not isomorphic to $V \otimes_{\mathbb{R}} V$ as vector spaces over $\mathbb{R}$ and compute their dimensions over $\mathbb{R}$.
(b) Let R be an integral domain and F its field of fractions. Prove that $F \otimes_R F \cong F \otimes_F F \cong F$ as F-modules. Note that the F-module structure on $F \otimes_R F$ is given by the extension of scalars construction (type I tensor product).

Problem 3. Let R be a commutative domain, and let M be a free R-module with basis $e_1, \ldots, e_k$. Prove that the element $e_1 \otimes e_2 + e_2 \otimes e_1 \in M \otimes M$ is not representable as a simple tensor $m \otimes n$ for some $m, n \in M$.

Problem 4 (practice). Let I and J be ideals of a (commutative) ring R, and let $\pi_I : R \to R/I$ and $\pi_J : R \to R/J$ be canonical projections.
(a) Prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor $\pi_I(1) \otimes \pi_J(r)$ for some $r \in R$.
(b) Prove that $R/I \otimes_R R/J \cong R/(I + J)$ (as R-modules).
(c) Show that there is a surjective R-module homomorphism $I \otimes_R J \to IJ$ such that $i \otimes j \mapsto ij$.
(d) Give an example where φ in (c) is not an isomorphism.

Problem 5. Let R be a commutative ring (with 1) and $n, m \in \mathbb{N}$. Prove that $R^n \otimes R^m \cong R^{nm}$ as R-algebras. As usual $R^k = \underbrace{R \oplus \ldots \oplus R}_{k \text{ times}}$.

Problem 6. The purpose of this problem is to classify 2-dimensional $\mathbb{R}$-algebras ($\mathbb{R}$=reals), that is, $\mathbb{R}$-algebras which are 2-dimensional as vector spaces over $\mathbb{R}$.
Let A be a 2-dimensional $\mathbb{R}$-algebra (as always, with 1).
(a) Let $u \in A$ be any element which is not an $\mathbb{R}$-multiple of 1. Prove that

(i) u generates A as an $\mathbb{R}$-algebra, that is, the minimal $\mathbb{R}$-subalgebra of A containing u and 1 is A itself.

(ii) u satisfies a quadratic equation $au^2 + bu + c = 0$ for some $a, b, c \in \mathbb{R}$ with $a \neq 0$.

(b) Show that there exists $v \in A$ such that $v^2 = -1$, $v^2 = 1$ or $v^2 = 0$. **Hint:** take any u as in (a), and look for v of the form $\alpha u + \beta$ with $\alpha, \beta \in \mathbb{R}$.

(c) Deduce from (b) that A is isomorphic as an $\mathbb{R}$-algebra to $\mathbb{R}[x]/(x^2 + 1)$, $\mathbb{R}[x]/(x^2 - 1)$ or $\mathbb{R}[x]/x^2$.

(d) Prove that the algebras $\mathbb{R}[x]/(x^2+1)$, $\mathbb{R}[x]/(x^2-1)$ and $\mathbb{R}[x]/x^2$ are pairwise non-isomorphic. **Hint:** this can be done with virtually no computations involved.