Homework Assignment # 2.

Plan for next week: Tensor products of modules, part 2 (10.4), algebras over commutative rings.

Problems, to be submitted by Thu, February 4th.

Problem 1[From Lecture 2]. Let R be a field, M and R-vector space, X a generating set of M, and Y a linearly independent subset of X. Prove that M has a basis B with $Y \subseteq B \subseteq X$.

Problem 2. Let R be a commutative domain, and let I be a non-principal ideal of R. Prove that I, considered as an R-module (with left-multiplication action) is indecomposable but not cyclic. **Hint:** One way to prove that I is indecomposable is to show that any two elements of I are linearly dependent over R.

Note: As we will prove in a couple of weeks, if R is a principal ideal domain, every finitely generated indecomposable module is cyclic.

Problem 3. Prove Schur's lemma [DF, problem 11, p.356].

Problem 4 (practice) [DF, problems 12-13, p.356].

Problem 5. Let R be a commutative ring. An R-module M is called *torsion* if for any $m \in M$ there exists nonzero $r \in R$ such that $rm = 0$. An R-module M is called *divisible* if for any nonzero $r \in R$ we have $rM = M$. In other words, M is divisible if for any $m \in M$ and nonzero $r \in R$ there exists $x \in M$ such that $rx = m$.

(a) Suppose that M is a torsion R-module and N is a divisible R-module. Prove that $M \otimes_R N = \{0\}$.

(b) Let $M = \mathbb{Q}/\mathbb{Z}$ considered as a $\mathbb{Z}$-module. Prove that $M \otimes_{\mathbb{Z}} M = \{0\}$.

Problem 6. Let R be a commutative ring, $\{N_\alpha\}$ a collection of R-modules and M another R-module.

(a) (practice, [DF, problem 14, p.376]) Prove that $M \otimes (\oplus N_\alpha) \cong \oplus(M \otimes N_\alpha)$ as R-modules (tensor products are over R).

(b) (see [DF, problem 15, p. 376]) Show by example that $M \otimes (\prod N_\alpha)$ need not be isomorphic to $\prod(M \otimes N_\alpha)$

Problem 7. Let $R \subseteq S$ be rings (not necessarily commutative), and let $R[x]$ (resp. $S[x]$) be the ring of polynomials over R (resp. S) (we assume that x commutes with everything). Prove that

$$S \otimes_R R[x] \cong S[x] \text{ as } S\text{-modules}$$