Homework Assignment # 10.

Plan for the next week:

Lecture 24 (April 20): Solvability of equation by radicals

Lecture 25 (April 22): Fundamental theorem of Algebra. Inverse Limits.

Problems, to be submitted by Thu, April 22nd.

Problem 1: Before doing this problem, read the first half of Section 14.4 in DF (pp. 591-593).

Definition 1: Let L/F be a finite separable extension and let $\overline{F}$ be an algebraic closure of F containing L. A subfield L' of $\overline{F}$ is called **conjugate to** L **over** F if $L' = \sigma(L)$ for some F-embedding of σ into $\overline{F}$. Note that L/F is Galois if and only if L does not have any F-conjugates besides L itself.

Definition 2: A finite extension K/F is called a p**-extension** if K/F is Galois and $Gal(K/F)$ is a p-group.

(a) Let L/F be a separable extension of degree n, and let K be the Galois closure of L over F. Prove that K can be written as a compositum $L_1 L_2 \dots L_n$ where $L_1, \dots L_n$ are (not necessarily distinct) conjugates of L over F.

(b) Let K/F and L/F be finite p-extensions. Prove that KL/F is also a p-extension.

(c) Suppose K/L and L/F are both p-extensions, and let M be the Galois closure of K over F (note: we do not know whether K/F is Galois or not). Prove that M/F is also a p-extension. **Hint:** use (a) and (b)

(d) Now assume only that L/F is a separable extension with $[L : F]$ a power of p, and let M be the Galois closure of L over F. Prove that $[M : F]$ need not be a power of p.

Problem 2: Let p and q be distinct primes with $q > p$, and let K/F be a Galois extension of degree pq. Prove that

(a) There exists a field L with $F \subseteq L \subseteq K$ and $[L : F] = q$

(b) There exists a unique field M with $F \subseteq M \subseteq K$ and $[M : F] = p$.

Problem 3: Let $f(x)$ and $g(x)$ be irreducible polynomials in $\mathbb{F}_p[x]$ of the same degree and let $F = \mathbb{F}_p[x]/(f(x))$. Prove that $g(x)$ splits completely over F.

Problem 4: DF, Problem 7 on p. 596.

Problem 5: Let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$, and as in class let $\mathbb{F}_{p^n} = \{x \in \overline{\mathbb{F}}_p : x^{p^n} = x\}$.

(a) Prove that $\overline{\mathbb{F}}_p = \cup_{n=1}^{\infty} \mathbb{F}_{p^n}$.

(b) Prove that the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ is uncountable (complete the argument outlined in class).

Problem 6: Prove the following analogue of Kummer's theorem for abelian extensions: Let $n \in \mathbb{N}$ and let F be a field containing primitive n^{th} root of unity.

(a) Let K/F be a finite Galois extension such that $\mathrm{Gal}(K/F)$ is abelian of exponent n. Then there exists $a_1, \ldots, a_t \in K$ s.t. $K = F(\sqrt[n]{a_1}, \ldots, \sqrt[n]{a_t})$, or more precisely, there exists $\alpha_1, \ldots, \alpha_t \in K$ s.t. $K = F(\alpha_1, \ldots, \alpha_t)$ and $\alpha_i^n \in F$ for all i.

(b) Conversely, suppose that $K = F(\sqrt[n]{a_1}, \ldots, \sqrt[n]{a_t})$ for some $a_1, \ldots, a_t \in F$. Prove that K/F is Galois, and $\mathrm{Gal}(K/F)$ is abelian of exponent n.

Hint: For (b) use one of the problems in the previous homework.