Homework Assignment # 1.

Plan for next week: 10.3 (generation of modules, free modules), start 10.4 (tensor products of modules).

Problems, to be submitted by Thu, January 28th.

Convention: All rings below are assumed to have 1, and all modules are left modules.

Problem 1: Let R be a commutative Noetherian ring and $\varphi : R \to R$ a surjective ring homomorphism. Prove that φ must be an isomorphism. **Hint:** Consider the ideals $\operatorname{Ker}(\varphi^n)$, $n \in \mathbb{N}$, where φ^n is φ composed with itself n times.

Problem 2: Let R be a commutative Noetherian ring. Prove that the ring $R[[x]]$ of power series over R is also Noetherian. **Hint:** As you may expect, this can be proved similarly to the Hilbert basis theorem (HBT) except that you have to consider the lowest degree terms, not the highest degree terms (which may not exist). In fact, the first part of the proof is even easier than in HBT, but you will need some kind of limit argument at the end.

Problem 3 (practice): Let M be an R-module for some ring R (not necessarily commutative).
(a) For a subset N of M the *annihilator of N in R* is defined to be the set $Ann_R(N) := \{r \in R : rn = 0 \text{ for any } n \in N\}$. Prove that $Ann_R(N)$ is a left ideal of R.
(b) Prove that if N is a submodule of M, then $Ann_R(N)$ is an ideal of R (that is, a two-sided ideal).
(c) For a subset I of R the *annihilator of I in M* is defined to be the set $Ann_M(I) := \{m \in M : xm = 0 \text{ for any } x \in I\}$. Find a natural condition on I which guarantees that $Ann_M(I)$ is a submodule of M.

Problem 4: Let R be a ring and let M be an R-module.
(a) Prove that for any $m \in M$, the map $x \mapsto xm$ from R to M is a homomorphism of R-modules (recall that R is an R-module with the left multiplication action).
(b) Assume that R is commutative, and let M be an R-module. Prove that $Hom_R(R, M) \cong M$ as R-modules. **Hint:** An element of $Hom_R(R, M)$ is uniquely determined by where it maps 1.

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Problem 5: In class we obtained a simple characterization of R-modules for $R = \mathbb{Z}$ and $R = F[x]$, with F a field. Formulate and prove similar characterizations for R-modules in the following cases:

(a) $R = \mathbb{Z}/n\mathbb{Z}$ for some $n \geq 2$; (b) $R = \mathbb{Z}[x]$; (c) $R = F[x, y]$, with F a field.

Problem 6 (practice): Let G be a group. The *integral group ring* $\mathbb{Z}[G]$ is defined as follows: as a set $\mathbb{Z}[G]$ is the collection of formal finite linear combinations of elements of G with integral coefficients, that is,

$$\mathbb{Z}[G] = \left\{ \sum_{g \in G} n_g g : n_g \in \mathbb{Z} \text{ and only finitely many } n_g \text{ are nonzero.} \right\}$$

Define the addition and multiplication on $\mathbb{Z}[G]$ by setting

$(\sum_{g \in G} n_g g) + (\sum_{g \in G} m_g g) = \sum_{g \in G} (n_g + m_g) g$ and

$(\sum_{g \in G} n_g g) \cdot (\sum_{g \in G} m_g g) = \sum_{g \in G} l_g g$ where $l_g = \sum_{h \in G} n_h m_{h^{-1} g}$ (note that this sum is finite).

In other words, multiplication in $\mathbb{Z}[G]$ is obtained by first setting $g \cdot h$ to be the product of g and h in G and then uniquely extending to arbitrary elements of $\mathbb{Z}[G]$ by distributivity.

Now the actual problem. Let M be an abelian group. Show that there is a natural bijection between $\mathbb{Z}[G]$-module structures on M and actions of G on M by group automorphisms (that is, actions of G on M such that for any $g \in G$ the map $m \mapsto gm$ is an automorphism of the abelian group M).

Problem 7: An R-module M is called *simple (or irreducible)* if M has no submodules besides $\{0\}$ and M. An R-module M is called *indecomposable* if M is not isomorphic to $N \oplus P$ for nonzero R-modules N and P.

(a) Prove that every simple module is indecomposable

(b) Describe all simple $\mathbb{Z}$-modules and all finitely generated indecomposable $\mathbb{Z}$-modules. Deduce that an indecomposable module need not be simple.

Problem 8: An R-module M is called *cyclic* if M is generated (as an R-module) by one element.

(a) Prove that cyclic R-modules are precisely the ones which are isomorphic to R/I for some left ideal I of R.

(b) Prove that every simple module is cyclic. Then show that simple R-modules are precisely the ones which are isomorphic to R/I for some maximal left ideal I of R.