Homework #9

Plan for next week: Basic properties of modules (10.1-10.3). Start tensor products of modules (10.4). See also Lectures 1-3 at

`http://people.virginia.edu/~mve2x/7752_Spring2010/`

Problems, to be submitted by 11:59pm on Sat, November 14th

1.

(a) Classify all abelian groups of order $360 = 2^3 \cdot 3^2 \cdot 5$ up to isomorphism. For each isomorphism type, state the corresponding elementary divisors form and invariant factors form.

(b) Let $n \in \mathbb{N}$, and decompose n as a product of primes: $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$. Find (with justification) the number of non-isomorphic abelian groups of order n. Express your answer in terms of the partition function P (where $P(n)$ is the number of partitions of n).

2. Let G be a finite abelian group. Prove that G is cyclic if and only if G does not contain a subgroup isomorphic to $B \oplus B$ for some non-trivial group B.

3.

Let G be an abelian group (not necessarily finitely generated), and let $\mathrm{Tor}(G)$ be the set of elements of finite order in G.

(a) Prove that $\mathrm{Tor}(G)$ is a subgroup. It is called the torsion subgroup of G.

(b) Prove that the quotient group $G/\mathrm{Tor}(G)$ is torsion-free, that is, $G/\mathrm{Tor}(G)$ has no elements of finite order apart from the identity element.

(b) For each prime p let $\mathrm{Tor}_p(G)$ be the set of elements of order p^k (with $k \geq 0$) in G. Prove that each $\mathrm{Tor}_p(G)$ is a subgroup of $\mathrm{Tor}(G)$ and that $\mathrm{Tor}(G) = \bigoplus_p \mathrm{Tor}_p(G)$ where p ranges over all primes.

Note: Recall that the definition of internal direct sum (of a potentially infinite collection of subgroups) was given at the end of Lecture 19. In the case of abelian groups written additively, this definition can be rephrased as follows:

Let $\{A_i\}_{i \in I}$ be a family of subgroups of A. Then $A = \bigoplus_{i \in I} A_i$ if

(1) $A = \langle A_i : i \in I \rangle$, that is (since A is abelian), every $a \in A$ can be written as a **finite** sum $a = a_1 + \ldots + a_m$ where each a_k lies in A_{i_k} for some $i_k \in I$

(2) for each $i \in I$ the intersection $A_i \cap \langle A_j : j \neq i \rangle$ is trivial. Since every element of $\langle A_j : j \neq i \rangle$ if a finite sum of elements of $\bigcup_{j \neq i} A_j$, this is the same as requiring that for any distinct indices $i, j_1, \ldots, j_m \in I$ the intersection $A_i \cap \langle A_{j_1}, \ldots, A_{j_m} \rangle$ is trivial.

4. Let X be a set. As in Lecture 21, define $FA(X)$ to be the group of all formal linear combinations $\sum\limits_{x \in X} \lambda_x x$ where each $\lambda_x \in \mathbb{Z}$ and only finitely many λ_x are nonzero. Let $F(X)^{ab} = F(X)/[F(X), F(X)]$ be the abelianization of $F(X)$, the free group on X.

In Lecture 22 we proved that $F(X)^{ab} \cong FA(X)$ by showing that both $F(X)^{ab}$ and $FA(X)$ are free objects on X in the category of abelian groups (and using the uniqueness of a free object up to isomorphism). Give another proof of the isomorphism $F(X)^{ab} \cong FA(X)$ by constructing homomorphisms in both directions and showing that they are mutually inverse.

5. Let p and q be primes with $p < q$ and $q \equiv 1 \mod p$, and let G be a non-abelian group or order pq. Recall that such G is unique up to isomorphism. Prove that G has a presentation

$$\langle x, y \mid x^p = 1, y^q = 1, xyx^{-1} = y^a \rangle$$

where a is coprime to q and the order of $[a]_q$ in $\mathbb{Z}_q^\times$ is equal to p.