Homework #8

Plan for next week: Classification of finitely generated abelian groups with applications (5.2). A very brief discussion of nilpotent and solvable groups (6.1). Free groups (6.3).

Problems, to be submitted by 11:59pm on Fri, October 30th

1. Let $n \geq 3$ be an integer and let S_n be the symmetric group on $\{1, 2, \ldots, n\}$. Let H be a subgroup of S_n with $[S_n : H] = n$. Prove that

$$H \cong S_{n-1}.$$

Hint: Start by constructing a suitable action of S_n associated to H. You may use the description of normal subgroups of S_n (the result for $n \geq 5$ follows very easily from the fact that A_n is simple).

(i) If $n \neq 4$, the only normal subgroups of S_n are S_n, A_n and $\{1\}$

(ii) The only normal subgroups of S_4 are S_4, A_4, V_4 (the Klein 4-group) and $\{1\}$

2. Let Ω be an infinite countable set (for simplicity you may assume that $\Omega = \mathbb{Z}$, the integers). Let $S(\Omega)$ be the group of all permutations of Ω. A permutation $\sigma \in S(\Omega)$ is called *finitary* if it moves only a finite number of points, that is, the set $\{i \in \Omega : \sigma(i) \neq i\}$ is finite. It is easy to see that finitary permutations form a subgroup of $S(\Omega)$ which will be denoted by $S_{fin}(\Omega)$. Finally, let $A_{fin}(\Omega)$ be the subgroup of even permutations in $S_{fin}(\Omega)$ (note that it makes sense to talk about even permutations in $S_{fin}(\Omega)$, but not in $S(\Omega)$).

(a) Prove that the group $A_{fin}(\Omega)$ is simple and that $A_{fin}(\Omega)$ is a subgroup of index two in $S_{fin}(\Omega)$. **Hint:** To prove the first assertion solve problem 5 in [DF, page 151].

(b) Prove that $A_{fin}(\Omega)$ and $S_{fin}(\Omega)$ are both normal in $S(\Omega)$.

(c) Prove that neither of the groups $S(\Omega)$ and $S_{fin}(\Omega)$ is finitely generated. **Hint:** The two groups are not finitely generated for completely different reasons.

(d) Construct a finitely generated subgroup G of $S(\Omega)$ which contains $S_{fin}(\Omega)$. **Note:** This example shows that a subgroup of a finitely generated group does not have to be finitely generated.

3. Let X be a non-empty set. Let $\{X_i\}_{i\in I}$ be a partition of X, that is, assume that each X_i is non-empty and $X = \sqcup_{i\in I} X_i$. The partition $\{X_i\}_{i\in I}$ is called **trivial** if either $|I| = 1$ (so that $X_i = X$ for the unique $i \in I$) or $|X_i| = 1$ for all $i \in I$; otherwise we will say that $\{X_i\}_{i\in I}$ is non-trivial.

Suppose now that a group G acts on X. A partition $\{X_i\}_{i\in I}$ of X will be called invariant (with respect to the given action) if for all $i \in I$ and $g \in G$ there exists $j = j(i,g) \in I$ such that $g(X_i) = X_j$. The action is called **primitive** if it does not admit any non-trivial invariant partition.

(a) Prove that if $|X| > 2$, then a primitive action must be transitive

(b) Prove that a 2-transitive action must be primitive

(c) Given $n \in \mathbb{N}$, let $[n] = \{1, 2, \dots, n\}$. Find (with proof) all n for which there exists an action of some G on $[n]$ which is transitive, but not primitive.

4. In each of the following cases determine (with proof) if a given group G decomposes as a semi-direct product $H \rtimes K$ or $K \rtimes H$ for a given subgroup H (if the answer is yes, explicitly describe K).

(a) $G = S_4$, $H = V_4 = \{e, (12)(34), (13)(24), (14)(23)\}$

(b) $G = GL_n(F)$ for some $n \in \mathbb{N}$ and a field F, $H = SL_n(F)$

(c) $G = SL_2(F)$ for some field F with $\mathrm{char}(F) \neq 2$, $H = Z(SL_2(F)) = \{diag(\lambda, \lambda) : \lambda = \pm 1\}$ (you can assume the second equality here without proof)

(d) G is the subgroup of $GL_n(F)$ consisting of all matrices $(a_{ij}) \in GL_n(F)$ with $a_{i1} = 1$ for $i > 1$ (in other words, G is the stabilizer of the line Fe_1 with respect to the standard action of G on F^n) and H is the subgroup consisting of all matrices $(a_{ij}) \in GL_n(F)$ with $a_{11} = 1$ and $a_{1i} = a_{i1} = 0$ for $i > 1$.