Homework #7

Plan for next week: Simple groups and proof of simplicity of A_n (4.6). Jordan-Hölder Theorem (3.4). Classification of finitely generated abelian groups with applications (5.2)

Problems, to be submitted by 11:59pm on Thu, October 22nd

1. DF, Problem 19 on page 131. **Note:** if G is a finite group, $x \in G$ and $\mathcal{K}(x)$ is the conjugacy class of x in G, then

$$|\mathcal{K}(x)| = [G : C_G(x)]$$

(this is just the orbit-stabilizer formula applied to the conjugation action of G on itself).

2.

(a) Prove Lemma 15.7 from class: Let H, K be groups, let φ and ψ be homomorphisms from K to $\mathrm{Aut}(H)$, and assume that there exists $\theta \in \mathrm{Aut}(K)$ such that $\varphi \circ \theta = \psi$. Prove that $H \rtimes_\varphi K \cong H \rtimes_\psi K$.

(b) DF, Problem 6 on page 184. **Note:** you will need to use a result from one of the first three homeworks.

3. The goal of this problem is to complete classification of groups of order 56 (we discussed the outline in Lecture 16).

(a) Let p be a prime, let P be a finite abelian p-group, and let H be a finite group with $p \nmid |H|$. Let φ_1 and φ_2 be homomorphisms from H to $\mathrm{Aut}(P)$, and let $G_i = P \rtimes_\varphi H$ for $i = 1, 2$. Suppose that $G_1 \cong G_2$. Prove that $\mathrm{Ker}\,(\varphi_1) \cong \mathrm{Ker}\,(\varphi_2)$. **Hint:** Compute the centralizer of P in $P \rtimes_\varphi H$.

(b) Let Q be a group of order 8, let $\Omega_4(Q)$ be the set of subgroups of Q isomorphic to $\mathbb{Z}_4$, and let $\Omega_{2,2}(Q)$ be the set of subgroups of Q isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Prove that $\mathrm{Aut}(Q)$ acts transitively on each of the sets $\Omega_4(Q)$ and $\Omega_{2,2}(Q)$. **Note:** Most likely you will need to argue separately for each isomorphism class.

(c) Now justify the entries in the right columns of the table on page 2 of class notes of Lecture 16 (they describe the number of isomorphic classes of groups of order 56 covered by the given case; you do not need to justify the entries for the number of homomorphisms). **Hint:** most likely you will need to use both parts of Problem 2, 3(a), 3(b), and the fact that a homomorphism from a given group G whose image has order ≤ 2 is completely determined by its kernel.

4. Redo Problem 4 on the first midterm (unless you got full credit). If you did not get a correct answer on the midterm, I encourage you to start with an easier problem – find the number of ideals of $\mathbb{Z}$ which contain 12 – and then generalize.

5. Let G be a group, let $H_1, \ldots, H_n$ be a finite collection of subgroups of G, and let $H_1 \times \ldots \times H_n$ denote their external direct product. Prove that the following three conditions are equivalent (note that (b) and (c) by themselves consist of several "subconditions"):

(a) The map $\varphi : H_1 \times \ldots \times H_n \to G$ given by

$$\varphi((h_1, \ldots, h_n)) = h_1 h_2 \ldots h_n$$

is a group isomorphism (the product on the right-hand side is taken in G)

(b) The following hold:
- (i) For any $i \neq j$ the subgroups H_i and H_j commute elementwise,
- (ii) For any i we have $H_i \cap H_i' = \{1\}$ where $H_i' = \langle H_j : j \neq i \rangle$, the subgroup generated by all H_j's besides H_i
- (iii) $G = \langle H_j : 1 \leq j \leq n \rangle$

(c) The following hold:
- (i) Each H_i is normal in G
- (ii) For any i we have $H_i \cap H_i' = \{1\}$ where $H_i' = \langle H_j : j \neq i \rangle$, the subgroup generated by all H_j's besides H_i
- (iii) $G = \langle H_j : 1 \leq j \leq n \rangle$