Homework #6

Plan for next week: Direct and semi-direct products and further applications of Sylow theorems (5.4, 5.5). Jordan-Hölder Theorem (3.4).

Problems, to be submitted by 11:59pm on Thu, October 15th

1. Let F be a finite field of order q.

(a) Prove that $|GL_n(F)| = \prod_{k=0}^{n-1}(p^n - p^k)$. **Note:** We outlined the argument in Lecture 14, so your main task is to carefully write down the details.

(b) Prove that $|SL_n(F)| = \frac{|GL_n(F)|}{q-1}$ for any $n \geq 2$.

2.

(a) Solve Problem 10 on page 117 in Dummit and Foote.

(b) Use Problem 10 to prove the index formula in Theorem 14.2 from class:

$$[G : K] = \sum_{x \in \Omega} [H : H \cap xKx^{-1}]$$

where Ω is a set of representatives of (H, K)-double cosets.

3. An action of a group G on a set X is called *transitive* if it has just one orbit, that is, for any $x, y \in X$ there exists $g \in G$ with $g.x = y$.

(a) Let $(G, X, .)$ be a group action. Prove that if $x, y \in X$ lie in the same orbit, then their stabilizers $Stab_G(x)$ and $Stab_G(y)$ are conjugate, that is, there exists $g \in G$ with $gStab_G(x)g^{-1} = Stab_G(y)$.

(b) Suppose that $(G, X, .)$ is a transitive action and fix $x \in X$. Prove that the kernel of this action is equal to $\bigcap_{g \in G} gStab_G(x)g^{-1}$

(c) Now suppose that G and X are both finite, $(G, X, .)$ is a transitive faithful action (where 'faithful' means the kernel is trivial) and G is abelian. Prove that for any $g \in G \setminus \{1\}$ the fixed set $Fix_X(g)$ is empty. Deduce that $|X| = |G|$. **Hint:** Use (b).

4. Let C be the cube in $\mathbb{R}^3$ whose vertices have coordinates $(\pm 1, \pm 1, \pm 1)$. Let G be the group of rotations of C, that is rotations in $\mathbb{R}^3$ which preserve the cube (you may assume that G is a group without proof). Let X be the set of 4 main diagonals of C (diagonals connecting the opposite vertices). Note that G naturally acts on X and therefore we have a homomorphism $\pi : G \to Sym(X) \cong S_4$. Prove that π is an isomorphism.

Hint: First show that G acts transitively on the 8 vertices of C. Then show that the stabilizer of a fixed vertex had order ≥ 3. This implies that $|G| \geq 24 = |S_4|$. Finally, show that π is injective (since $|G| \geq |S_4|$, this would force π to be an isomorphism).

5. Let G be a finite group, P a Sylow p-subgroup of G for some p and H a subgroup of G such that $N_G(P) \subseteq H \subseteq G$. Prove that $N_H(P) = N_G(P)$ and $[G : H] \equiv 1 \mod p$.

6. Prove that a group of order $132 = 3 \cdot 4 \cdot 11$ has a normal Sylow 11-subgroup. **Hint:** first show that G has a normal Sylow p-subgroup for some $p \in 2, 3, 11$.

7. Explicitly describe a Sylow p-subgroup of S_{p^2}. See next page for a hint.

Hint for 7: The largest power of p dividing $(p^2)!$ is p^{p+1}. First find a subgroup of order p^p in S_{p^2} (this is quite easy) and then think how to enlarge it to a subgroup of order p^{p+1}. Proposition 10 on p. 125 of DF is very relevant here.