Homework #4

Plan for next week: Hilbert's Nullstellensatz (§ 15.3). Start Group Theory. I plan to review isomorphism theorems (§ 3.3), briefly talk about the commutator subgroups, automorphism groups (§ 4.4) and possibly the classification theorem for finitely generated abelian groups (§ 5.2).

Problems, to be submitted by 11:59pm on Thu, September 24th

1. Give an example of a domain R (other than a field or the zero ring) which has no irreducible elements. **Hint:** Start with the ring of power series $R = F[[x]]$ where F is a field. Then up to associates x is the only irreducible element of R. Construct a larger ring $R_1 \supseteq R$ s.t. x is reducible in R_1, but $R_1 \cong F[[x]]$. Then iterating the process construct an infinite ascending chain $R \subseteq R_1 \subseteq R_2 \subseteq \ldots$ and consider its union.

2. Let R be a commutative Noetherian ring and $\varphi : R \to R$ a surjective ring homomorphism. Prove that φ must be an isomorphism. **Hint:** Consider the ideals $\mathrm{Ker}(\varphi^n)$, $n \in \mathbb{N}$, where φ^n is φ composed with itself n times.

3. Let I be an ideal of $\mathbb{Z}[x]$, and suppose that I contains a monic polynomial $f(x)$ of degree n. Prove that I can be generated (as an ideal) by at most $n+1$ elements.

4. DF, Problem 19 on p. 332. Make sure to read about the Buchberger's algorithm in 9.6 prior to solving this problem.

5. Let k be a field. Recall that an algebraic set $V \subseteq k^n$ is called irreducible if $V \neq \emptyset$ and V cannot be written as the union $V = V_1 \cup V_2$ where V_1 and V_2 are both algebraic, with $V_1 \neq V$ and $V_2 \neq V$.

(a) (practice) Prove that V is irreducible if and only if its vanishing ideal $I(V)$ is prime.

(b) It is not hard to prove that any algebraic set V can be uniquely written as a union of finitely many algebraic subsets

$$V = \cup_{i=1}^{k} V_i$$

where V_i's are irreducible and do not contain each other (you can assume this as a fact). Such V_i's are called the **irreducible components of V** .

Assume that k is infinite, and let

$$V = Z(xy - y, x^2z - z) \subset k^3,$$

the set of common zeroes of $xy-y$ and x^2z-z. Find irreducible components of V and their vanishing ideals (with justification!). The answer will depend on $char(k)$, the characteristic of k.

6. Let k be a field and $n \in \mathbb{N}$. The **Zariski topology** on k^n is the unique topology in which closed subsets are precisely the algebraic subsets of k^n.

(a) Check the topology axioms for the Zariski topology, that is prove that,
  (i) If V and W are algebraic, then $V \cup W$ is algebraic
  (ii) If $\{V_i\}_{i \in I}$ is any collection of algebraic subsets, then $\bigcap_{i \in I} V_i$ is also algebraic

(b) Prove that $Z(I(Y)) = \overline{Y}$ for every $Y \subseteq k^n$ where $\overline{Y}$ is the closure of Y in the Zariski topology.

In Problem 7 we identify the set $Mat_n(k)$ of $n \times n$ matrices over a field k with k^{n^2} and thus can talk about Zariski topology on $Mat_n(k)$.

Problem 7: Let k be a field and $n \in \mathbb{N}$. Determine if the following subsets of $Mat_n(k)$ are Zariski closed and prove your answer.

(a) $SL_n(k) = \{A \in Mat_n(k) : \det(A) = 1\}$.
(b) $GL_n(k) = \{A \in Mat_n(k) : \det(A) \neq 0\}$.
(c) The set of all matrices in $Mat_n(k)$ which have rank $\leq d$ for some fixed $1 \leq d \leq n$ (the answer may be different for different d).