Homework #3

Plan for next week: Finite fields (§ 9.5), Hilbert's basis Theorem (§ 9.6).

Problems, to be submitted by 11:59pm on Thu, September 17th

1. Let R and S be rings with 1, and let $\varphi : R \to S$ be a ring homomorphism such that $\varphi(1_R) = 1_S$.

(a) Prove that $\varphi(R^\times) \subseteq S^\times$

(b) Give an example where φ is surjective, but $\varphi(R^\times) \neq S^\times$.

2. Let $m, n \in \mathbb{N}$ with $m \mid n$, and define $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ by

$$f(x + n\mathbb{Z}) = x + m\mathbb{Z}$$

(note that f is well defined precisely because $m \mid n$). Prove that the associated map of the groups of units $f : (\mathbb{Z}/n\mathbb{Z})^\times \to (\mathbb{Z}/m\mathbb{Z})^\times$ is surjective.

Hint: First prove this directly for special values of n and m (using the standard characterization of units in $\mathbb{Z}/k\mathbb{Z}$ from HW#1.3). Then use another homework problem to do the general case.

3. Use HW#2.5 to find all maximal ideals of $\mathbb{Z}[x]$ (the ring of polynomials over $\mathbb{Z}$ in one variable) which contain $x^2 + 1$ and 15. You will need to use standard results about primes (=irreducible) elements in $\mathbb{Z}[i]$ – see the corresponding section in DF (pp. 289-292).

4. Let $R = \mathbb{Z}[\sqrt{5}] = \{a + b\sqrt{5} : a, b \in \mathbb{Z}\}$. Find an element of R which is irreducible but not prime and deduce that R is not a unique factorization domain (UFD).

Hint: Consider the equality $2 \cdot 2 = (\sqrt{5} + 1)(\sqrt{5} - 1)$. In order to check whether some element of R is irreducible it is convenient to use the standard norm function $N : R \to \mathbb{Z}_{\geq 0}$ given by $N(a+b\sqrt{5}) = |(a+b\sqrt{5})(a - b\sqrt{5})| = |a^2 - 5b^2|$ (just as in the $\mathbb{Z}[\sqrt{3}]$ example considered in class, it is easy to check that $N(uv) = N(u)N(v)$).

5. Let $R = \mathbb{Z} + x\mathbb{Q}[x]$, the subring of $\mathbb{Q}[x]$ consisting of polynomials whose constant term is an integer.

(a) Show that the element αx, with $\alpha \in \mathbb{Q}$ is NOT irreducible in R. Then show that x cannot be written as a product of irreducibles in R. Note that by Lecture 5, this implies that R is not Noetherian.

(b) Now prove directly that R is not Noetherian by showing that $I = x\mathbb{Q}[x]$ is an ideal of R which is not finitely generated.

(c) Give an example of a non-Noetherian domain which is a UFD.

6. Let R be a commutative ring with 1.

(a) Let M be an ideal of R. Prove that the following conditions are equivalent:

(i) M is the unique maximal ideal of R

(ii) every element of $R \setminus M$ is invertible.

Rings satisfying these equivalent conditions are called local.

(b) Let F be a field and $F[[x]]$ the ring of power series over F. Prove that $F[[x]]$ is local.

(c) Now let R be arbitrary, let P be a prime ideal of R and $S = R \setminus P$. Prove that the localization $S^{-1}R$ is local and explicitly describe its unique maximal ideal.