Homework #2

Plan for next week: PIDs, continued (§ 8.2), UFDs (§ 8.3), polynomial rings over UFDs (§ 9.3).

Problems, to be submitted by 11:59pm on Thu, September 10th

1. Let R be a finitely generated ring (not necessarily with 1). Use Zorn's lemma to show that R has a maximal subring (by definition a maximal subring is a maximal element of the set of proper subrings of R partially ordered by inclusion). Give a detailed argument.

2. Let R be a commutative ring with 1. The *nilradical* of R denoted $Nil(R)$ is the set of all nilpotent elements of R, that is

$$Nil(R) = \{a \in R : a^n = 0 \text{ for some } n \in \mathbb{N}\}.$$

Prove that

(a) $Nil(R)$ is an ideal of R

(b) $Nil(R)$ is contained in every prime ideal of R. Later we will show that $Nil(R)$ is equal to the intersection of all prime ideals.

3. Let R be a commutative ring with 1, and let D be a subset of R which is closed under multiplication such that $1 \in D$ and $0 \notin D$.

(a) Prove that the relation $\sim$ in the definition of rings of fractions RD^{-1} is an equivalence relation

(b) The operations $+$ and $\cdot$ on RD^{-1} (given by $\frac{r_1}{d_1} + \frac{r_2}{d_2} = \frac{r_1 d_2 + r_2 d_1}{d_1 d_2}$ and $\frac{r_1}{d_1} \cdot \frac{r_2}{d_2} = \frac{r_1 r_2}{d_1 d_2}$) are well defined.

4. Let $R = \mathbb{Z}_{14}$, $D = \{\bar{1}, \bar{2}, \bar{4}, \bar{8}\}$ (note that D is multiplicatively closed but it does contain zero divisors). Prove that the localization RD^{-1} is isomorphic to $\mathbb{Z}_7$. **Hint:** What can you say about the map $\iota : R \to RD^{-1}$ in this case?

5. Let $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$ be the ring of Gaussian integers.

(a) Prove that $\mathbb{Z}[i]$ is a Euclidean domain.

(b) Prove that $\mathbb{Z}[i] \cong \mathbb{Z}[x]/(x^2 + 1)$

6. Let D be a positive integer such that $D \equiv 3 \mod 4$, and let $R = \mathbb{Z}[\frac{1+\sqrt{-D}}{2}]$, that is, R is the minimal subring of $\mathbb{C}$ containing $\mathbb{Z}$ and $\frac{1+\sqrt{-D}}{2}$.

(a) Prove that $R = \{a + b\frac{1+\sqrt{-D}}{2} : a, b \in \mathbb{Z}\}$. It should be clear from your argument where the assumption $D \equiv 3 \mod 4$ is used (otherwise the result is simply not true).

(b) Assume that $D = 3, 7$ or 11. Prove that R is a Euclidean domain. **Hint:** It is important to use the right Euclidean norm N. Before claiming that your N is multiplicative, better check it is actually true. Your argument should "barely" work for $D = 11$; if it still "works" for $D = 15$, it is a wrong argument.