Homework #10

Plan for next week: Tensor products of modules and algebras (10.4). See also Lectures 3-5 at

`http://people.virginia.edu/~mve2x/7752_Spring2010/`

Problems, to be submitted by 11:59pm on Sat, November 21st

Convention: All rings below are assumed to have 1, and all modules are left modules.

1. Problem #4 from Midterm #2 (no need to submit if you received full credit on the midterm).

2. Let R be a ring and let M be an R-module.

(a) Prove that for any $m \in M$, the map $x \mapsto xm$ from R to M is a homomorphism of R-modules (recall that R is an R-module with the left multiplication action).

(b) Assume that R is commutative, and let M be an R-module. Prove that $Hom_R(R, M) \cong M$ as R-modules. **Note:** For the definition and justification of the R-module structure on the set $Hom_R(M, N)$ (where R is commutative and M and N are R-modules) see Proposition 2 on page 346 in DF. **Hint:** An element of $Hom_R(R, M)$ is uniquely determined by where it maps 1.

3. An R-module M is called *simple (or irreducible)* if M has no submodules besides $\{0\}$ and M. An R-module M is called *indecomposable* if M is not isomorphic to $N \oplus P$ for nonzero R-modules N and P.

(a) Prove that every simple module is indecomposable

(b) Describe (with proof) all simple $\mathbb{Z}$-modules and all finitely generated indecomposable $\mathbb{Z}$-modules. Deduce that an indecomposable module need not be simple.

4. An R-module M is called *cyclic* if M is generated (as an R-module) by one element.

(a) Prove that cyclic R-modules are precisely the ones which are isomorphic to R/I for some left ideal I of R.

(b) Prove that every simple module is cyclic. Then show that simple R-modules are precisely the ones which are isomorphic to R/I for some maximal left ideal I of R.

5. Let R be a commutative domain, and let I be a non-principal ideal of R. Prove that I, considered as an R-module (with left-multiplication action) is indecomposable but not cyclic. **Hint:** One way to prove that I is indecomposable is to show that any two elements of I are linearly dependent over R. **Note:** As we will prove in Algebra-II, if R is a principal ideal domain, every finitely generated indecomposable module is cyclic.

6. Let R be a commutative ring. An R-module M is called *torsion* if for any $m \in M$ there exists nonzero $r \in R$ such that $rm = 0$. An R-module M is called *divisible* if for any nonzero $r \in R$ we have $rM = M$. In other words, M is divisible if for any $m \in M$ and nonzero $r \in R$ there exists $x \in M$ such that $rx = m$.

(a) Suppose that M is a torsion R-module and N is a divisible R-module. Prove that $M \otimes_R N = \{0\}$.

(b) Let $M = \mathbb{Q}/\mathbb{Z}$ considered as a $\mathbb{Z}$-module. Prove that $M \otimes_{\mathbb{Z}} M = \{0\}$.

7. Let R be a commutative ring, and let M be a free R-module with basis $e_1, \ldots, e_k$. Prove that the element $e_1 \otimes e_2 + e_2 \otimes e_1 \in M \otimes M$ is not representable as a simple tensor $m \otimes n$ for some $m, n \in M$. **Note:** You may want to start with the case where R is a domain (where the proof is a little bit easier) and then think how to modify the argument in the general case.