Homework #9.

Plan for next week: Unique Factorization Domains (8.3, 9.3) and Irreducibility criteria in polynomial rings (9.4). Note that we already discussed some of the main results from 8.3.

Problems, to be submitted by Thursday, November 7th

1. Let $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$ be the ring of Gaussian integers.

(a) Prove that $\mathbb{Z}[i]$ is a Euclidean domain.
(b) Prove that $\mathbb{Z}[i] \cong \mathbb{Z}[x]/(x^2 + 1)$
(c) Now use (a) and (b) to find all maximal ideals of $\mathbb{Z}[x]$ (the ring of polynomials over $\mathbb{Z}$ in one variable) which contain $x^2 + 1$ and 15. If you have not studied Gaussian integers before, carefully read the corresponding section in DF (pp. 289-292).

2. Let D be a positive integer such that $D \equiv 3 \mod 4$, and let $R = \mathbb{Z}[\frac{1+\sqrt{-D}}{2}]$, that is, R is the minimal subring of $\mathbb{C}$ containing $\mathbb{Z}$ and $\frac{1+\sqrt{-D}}{2}$.

(a) Prove that $R = \{a + b\frac{1+\sqrt{-D}}{2} : a, b \in \mathbb{Z}\}$. You may skip details, but it should be clear from your argument where the assumption $D \equiv 3 \mod 4$ is used (otherwise the result is simply not true).
(b) Assume that $D = 3, 7$ or 11. Prove that R is a Euclidean domain.

3. Let $R = \mathbb{Z}[\sqrt{5}] = \{a + b\sqrt{5} : a, b \in \mathbb{Z}\}$. Find an element of R which is irreducible but not prime and deduce that R is not a unique factorization domain (UFD).

Hint: Consider the equality $2 \cdot 2 = (\sqrt{5}+1)(\sqrt{5}-1)$. In order to check whether some element of R is irreducible it is convenient to use the standard norm function $N : R \to \mathbb{Z}_{\geq 0}$ given by $N(a+b\sqrt{5}) = |a^2 - 5b^2|$ (note that $N(uv) = N(u)N(v)$).

4. Let $R = \mathbb{Z} + x\mathbb{Q}[x]$, the subring of $\mathbb{Q}[x]$ consisting of polynomials whose constant term is an integer.

(a) Show that the element αx, with $\alpha \in \mathbb{Q}$ is NOT irreducible in R. Then show that x cannot be written as a product of irreducibles in R. Note that by Lecture 18, this implies that R is not Noetherian.

(b) Now prove directly that R is not Noetherian by showing that $I = x\mathbb{Q}[x]$ is an ideal of R which is not finitely generated.

(c) Give an example of a non-Noetherian domain which is a UFD.

5. Let R be a commutative ring with 1.

(a) Let M be an ideal of R. Prove that the following conditions are equivalent:

(i) M is the unique maximal ideal of R

(ii) every element of $R \setminus M$ is invertible.

Rings satisfying these equivalent conditions are called local.

(b) Let F be a field and $F[[x]]$ the ring of power series over F. Prove that $F[[x]]$ is local.

(c) Now let R be arbitrary, let P be a prime ideal of R and $S = R \setminus P$. Prove that the localization $S^{-1}R$ is local and explicitly describe its unique maximal ideal.