Homework #8.

Plan for next week: Chapter 8 (Euclidean domains, principal ideal domains and unique factorization domains).

Problems, to be submitted by Thursday, Oct 31

1. (a) Let G be a finitely generated group. Use Zorn's lemma to show that G has a maximal subgroup (recall that a maximal subgroup of G is a maximal element of the set of proper subgroups of G partially ordered by inclusion). **Hint:** The key step is to show that if $\mathcal{C}$ is a chain of proper subgroups of G, then the union of subgroups in this chain is not the whole G.
(b) (optional) If your proof in (a) is correct, a nearly identical argument should imply that G always has a maximal normal subgroup. However, the latter is true under weaker assumptions on G. Can you find a natural condition on G (weaker than finite generation) that guarantees the existence of a maximal normal subgroup? Can you give an example of a group which has a maximal normal subgroup, but no maximal subgroup?

2. Let R be a commutative ring with 1. The *nilradical* of R denoted $Nil(R)$ is the set of all nilpotent elements of R, that is

$$Nil(R) = \{a \in R : a^n = 0 \text{ for some } n \in \mathbb{N}\}.$$

The *Jacobson radical* of R denoted by $J(R)$ is the intersection of all maximal ideals of R. Prove that

(a) $Nil(R)$ and $J(R)$ are ideals of R
(b) $Nil(R) \subseteq J(R)$.

3. (a) Problem 7.3.34. Note: in all exercises in 7.3 R is assumed to be a ring with 1 (this is crucial for this problem). Also note that IJ is NOT defined to be the set $\{ij : i \in I, j \in J\}$; by definition, IJ is the set of finite sums of elements of the form ij, with $i \in I, j \in J$.
(b) Read the section on the Chinese remainder theorem (7.6).

4. Let m and n be positive integers with $m \mid n$, and let $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ be the natural projection. Prove that the associated map of the groups of units $f : (\mathbb{Z}/n\mathbb{Z})^\times \to (\mathbb{Z}/m\mathbb{Z})^\times$ is surjective.

5. Let $n \in \mathbb{N}$ and consider $\mathbb{Z}^n$ as a ring with component-wise addition and multiplication (note that $\mathbb{Z}^n$ considered as a group is precisely the additive group of this ring). Let $\text{Aut}_{\text{ring}}(\mathbb{Z}^n)$ be the group of ring automorphisms of $\mathbb{Z}^n$. Prove that $\text{Aut}_{\text{ring}}(\mathbb{Z}^n) \cong S_n$. **Note:** recall that $\text{Aut}_{\text{group}}(\mathbb{Z}^n) \cong GL_n(\mathbb{Z})$, as discussed in Lecture 2.

6. Let $R = \mathbb{Z}_{14}$, $D = \{\bar{1}, \bar{2}, \bar{4}, \bar{8}\}$ (note that D is multiplicatively closed but it does contain zero divisors). Prove that the localization RD^{-1} is isomorphic to $\mathbb{Z}_7$.