Homework #7.

Plan for next week: We will start ring theory. I plan to give a brief survey of Chapter 7, with emphasis on the following topics: maximal ideals (7.4), rings of fractions and localizations (7.5) and Chinese Remainder Theorem (7.6).

Problems, to be submitted by Thursday, October 24th

1. Let $G = \mathrm{SL}_2(\mathbb{Z})$, and let $X = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, and consider the natural action of G on $\mathbb{Z}^2$. Find subsets A and B of $\mathbb{Z}^2$ satisfying the hypotheses of the Ping-Pong Lemma (with respect to this action). This implies that $\langle X, Y \rangle$ is free of rank 2.

2. Let G be a group, S a subset of G and $H = \langle S \rangle$ the subgroup generated by S. One says that H is *freely generated by S* if the evaluation homomorphism $\varepsilon : F(S) \to H$ is an isomorphism.

Now let $G = F(a, b)$, the free group with 2 generators a and b. For each $n \in \mathbb{N}$ let $s_n = a^n b a^{-n} \in G$, and let $S = \{s_n\}_{n=1}^{\infty}$. Prove that $\langle S \rangle$ is free of countable rank (this gives another example of an infinitely generated subgroup of a finitely generated group). Do not use the theorem that subgroups of free groups are free. Give a detailed argument (what I have in mind is a purely combinatorial argument).

3. Given a group G, the quotient $G/[G, G]$ is called the *abelianization* of G and denoted by G^{ab} (basic properties of the commutator subgroup imply that G^{ab} is the largest abelian quotient of G).

(a) Let $n \in \mathbb{N}$. Prove that $F_n^{ab} \cong \mathbb{Z}^n$.
(b) Use (a) to prove that if X and Y are finite sets such that $F(X) \cong F(Y)$, then $|X| = |Y|$ (the corresponding result for arbitrary X and Y can be deduced similarly from a suitable generalization of (a)).

4. Let X be a set and $F(X)$ the free group on X. Given $f \in F(X)$, the *length of f*, denoted by $l(f)$ is defined to be the length of the unique reduced word in $X \cup X^{-1}$ representing f. Equivalently, $l(f)$ is the smallest n such that $f = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$ for some $x_i \in X$ and $\varepsilon_i \in \{\pm 1\}$.

(a) Prove that for any $f \in F(X)$ there exist integers $a, b \in \mathbb{Z}_{\geq 0}$ (depending on f) such that $l(f^n) = na + 2b$ for any $n \in \mathbb{N}$.

Describe explicitly (i.e. give an algorithm) how to compute a and b for a given f.

(b) Use (a) to show that free groups are torsion-free.

5.

(a) Explain why for any $n \in \mathbb{N}$ there are only finitely many isomorphism classes of groups of order n.

(b) Let G be a finitely generated group and H a finite group. Prove that there are only finitely many homomorphisms from G to H.

(c) Let G be a finitely generated group. Prove that for any $n \in \mathbb{N}$ there are only finitely many normal subgroups of index n in G. Then deduce that G has only finitely many subgroups of index n (use the small index lemma).

6. Let p and q be primes with $p < q$ and $q \equiv 1 \mod p$, and let G be a non-abelian group or order pq. Recall that such G is unique up to isomorphism. Prove that G has a presentation $\langle x, y \mid x^p = 1, y^q = 1, xyx^{-1} = y^a \rangle$ where a is coprime to q and the order of $\bar{a}$ in $\mathbb{Z}_q$ is equal to p.

Hint: Let $\widehat{G} = \langle x, y \mid x^p = 1, y^q = 1, xyx^{-1} = y^a \rangle$. By definition $\widehat{G}$ is the quotient of $F(x, y)$ by the normal closure of the set $\{x^p, y^q, xyx^{-1}y^{-a}\}$. To prove the statement first show that G has elements X and Y satisfying the above 3 relations; then show that there is a surjective homomorphism $\phi : \widehat{G} \to G$ such that $\phi(x) = X$ and $\phi(y) = Y$. Finally, prove that $|\widehat{G}| \leq pq$ and deduce that ϕ is an isomorphism.