Homework #6.

Plan for next week: Free groups and presentations of groups by generators and relators (§ 6.3).

Problems, to be submitted by Thursday, October 17th

1. Let $G_1, \ldots, G_k$ be non-abelian simple groups and let $G = G_1 \times \ldots \times G_k$. Prove that every normal subgroup of G is equal to $H_1 \times \ldots \times H_k$ where for each i either $H_i = G_i$ or $H_i = \{1\}$. You do not have to turn in this problem if you got full credit for 2(b) on the take-home part of Midterm 1.

2. Let G be a group.

(a) Prove that if N is a normal subgroup of G, then for any $k \in \mathbb{N}$ we have $\gamma_k(G/N) = (\gamma_k G \cdot N)/N$.

(b) Now assume that G is nilpotent of class $c \geq 1$. Prove that $G/Z(G)$ is nilpotent of class exactly $c-1$ (we proved inequality in one direction in class).

3. (a) Let R be an associative ring with 1, and let $a, b \in R$ be such that $1+a$ and $1+b$ are invertible. Prove the following formula

$$(1+a)^{-1}(1+b)^{-1}(1+a)(1+b) = 1 + (1+a)^{-1}(1+b)^{-1}(ab - ba).$$

(b) Let R be an associative ring with 1 and $n \geq 2$ be an integer, and let $U_n(R)$ be the upper unitriangular subgroup of $GL_n(R)$. Prove that $U_n(R)$ is nilpotent of class $n-1$ (we briefly outlined the proof in class). Note that you will need to apply (a) not to R itself but to the ring of $n \times n$ matrices over R.

4. Problems 24 and 25 on page 199 of DF.

5. Problems 31 and 32 on page 200 of DF. **Note:** Problem 31 follows very easily from the lemma about the structure of minimal normal subgroups proved in class.

6. (a) Prove that a Sylow p-subgroup of S_{p^2} is isomorphic to $\mathbb{Z}_p \, wr \, \mathbb{Z}_p$.

(b) Prove that if A and B are solvable groups, then $A \, wr \, B$ is also solvable.

(c) (bonus) Find all integers $m, n \geq 2$ for which $\mathbb{Z}_m \, wr \, \mathbb{Z}_n$ is nilpotent.