Homework #5, due on Fri, Oct 11th (by 1pm in my mailbox).

Plan for the next week: Nilpotent and solvable groups (the best approximation in Dummit and Foote is 6.1, but we will not follow it very closely).

1. Solve Problems 1(c) and 3(b) from the in-class part of Midterm 1 (you do not have to redo problems for which you got full credit on the midterm):

(1c) Let p be a prime. Prove that any group of order p^2 is abelian.

(3b) Let p be an odd prime. Find the smallest n for which S_n contains a subgroup of order $2p$.

2.

(a) Classify all abelian groups of order $360 = 2^3 \cdot 3^2 \cdot 5$ up to isomorphism. For each isomorphism type, state the corresponding elementary divisors form and invariant factors form.

(b) Let $n \in \mathbb{N}$, and decompose n as a product of primes: $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$. Find (with justification) the number of non-isomorphic abelian groups of order n. Express your answer in terms of the partition function P (where $P(n)$ is the number of partitions of n).

3. Let G be a finite abelian group. Prove that G is cyclic if and only if G does not contain a subgroup isomorphic to $B \oplus B$ for some non-trivial group B.

4. Given a finite group G and a positive integer n, denote by $a_n(G)$ the number of elements of G of order n and by $b_n(G)$ the number of elements of G of order dividing n. The goal of this problem is to prove the following theorem:

Theorem A: If G and H are finite abelian groups and $a_n(G) = a_n(H)$ for all n, then G is isomorphic to H.

(a) Let G and H be finite groups. Prove that $a_n(G) = a_n(H)$ for all $n \iff b_n(G) = b_n(H)$ for all n.

(b) Suppose that $G = X \times Y$. Prove that $b_n(G) = b_n(X) b_n(Y)$.

(c) Suppose that G and H are finite abelian groups s.t. $a_n(G) = a_n(H)$ for all n. Prove that there exists a non-trivial group C s.t. $G \cong A \times C$ and $H \cong B \times C$ for some groups A and B. **Hint:** Use the classification theorem in invariant factors form.

(d) Now use (a),(b) and (c) and induction to prove Theorem A.

5. Let G be an abelian group (not necessarily finitely generated), and let $Tor(G)$ be the set of elements of finite order in G. Recall that $Tor(G)$ is a subgroup of G (since G is abelian), called the torsion subgroup of G.

(a) Prove that the quotient group $G/Tor(G)$ is torsion-free.

(b) For each prime p let $Tor_p(G)$ be the set of elements of order p^k (with $k \geq 0$) in G. Prove that each $Tor_p(G)$ is a subgroup of $Tor(G)$ and that $Tor(G) = \bigoplus_p Tor_p(G)$ where p ranges over all primes.

Note: Recall that if A is an abelian group written additively and $\{A_i\}_{i \in I}$ is a family of its subgroups, then $A = \bigoplus_{i \in I} A_i$ means that

(1) $A = \langle A_i : i \in I \rangle$, that is (since A is abelian), every $a \in A$ can be written as a finite! sum $a = a_1 + \ldots + a_m$ where each a_k lies in A_{i_k} for some $i_k \in I$

(2) for each $i \in I$ the intersection $A_i \cap \langle A_j : j \neq i \rangle$ is trivial. Since every element of $\langle A_j : j \neq i \rangle$ if a finite sum of elements of $\bigcup_{j \neq i} A_j$, this is the same as requiring that for any distinct indices $i, j_1, \ldots, j_m \in I$ the intersection $A_i \cap \langle A_{j_1}, \ldots, A_{j_m} \rangle$ is trivial.

6.

(a) Let A and B be finitely generated groups. Prove that the restricted wreath product $A \wr B$ is also finitely generated. **Hint:** Recall that $AwrB = C \rtimes B$ where $C = \oplus_{b \in B} A_b$ (with each $A_b \cong A$). Let S be a generating set for A, T a generating set for B, fix $b \in B$, and let S_b be the image of S under an isomorphism $A \to A_b$. Prove that $S_b \cup T$ generates $AwrB$.

(b) Use (a) to give a simple example showing that a subgroup of a finitely generated group may not be finitely generated.