Homework #4.

Plan for next week: Simplicity of A_n (4.6), classification of finitely generated abelian groups (5.2).

Problems, to be submitted by Thursday, September 26th

1.

(a) Prove Observation 8.2 from class: Let H, K be groups, let ϕ and ψ be homomorphisms from K to $\mathrm{Aut}(H)$, and assume that there exists $\theta \in \mathrm{Aut}(K)$ such that $\phi \circ \theta = \psi$. Prove that $H \rtimes_\phi K \cong H \rtimes_\psi K$.

(b) DF, Problem 6 on page 184.

2. DF, Problem 7(a)(c)(e) on page 185. Note that we proved (b) in class (Lecture 6). Clarification for part (c): for each isomorphism class of S you are asked to construct a certain number of non-isomorphic groups of order 56 with normal 7-Sylow and 2-Sylow isomorphic to S. You are not asked to prove that your groups cover all possible isomorphism classes (this part is optional and can be done using Problem 2(a) above). If you are using the hint in brackets following part (c), you should prove the statement in the hint.

3. Let $n \geq 3$ be an integer and let S_n be the symmetric group on $\{1, 2, \ldots, n\}$. Let H be a subgroup of S_n with $[S_n : H] = n$. Prove that

$$H \cong S_{n-1}.$$

Hint: Start by constructing a suitable action of S_n associated to H. You may use the description of normal subgroups of S_n which will be proved in class next Tuesday:

(i) If $n \neq 4$, the only normal subgroups of S_n are S_n, A_n and $\{1\}$

(ii) The only normal subgroups of S_4 are S_4, A_4, V_4 (the Klein 4-group) and $\{1\}$

4. Let Ω be an infinite countable set (for simplicity you may assume that $\Omega = \mathbb{Z}$, the integers). Let $S(\Omega)$ be the group of all permutations of Ω. A permutation $\sigma \in S(\Omega)$ is called *finitary* if it moves only a finite number of points, that is, the set $\{i \in \Omega : \sigma(i) \neq i\}$ is finite. It is easy to see that finitary permutations form a subgroup of $S(\Omega)$ which will be denoted by $S_{fin}(\Omega)$. Finally, let $A_{fin}(\Omega)$ be the subgroup of even permutations in $S_{fin}(\Omega)$ (note that it makes sense to talk about even permutations in $S_{fin}(\Omega)$, but not in $S(\Omega)$).

(a) Prove that the group $A_{fin}(\Omega)$ is simple and that $A_{fin}(\Omega)$ is a subgroup of index two in $S_{fin}(\Omega)$. **Hint:** To prove the first assertion solve problem 5 in [DF, page 151]. You may use the fact that A_n is simple for $n \geq 5$.

(b) Prove that $A_{fin}(\Omega)$ and $S_{fin}(\Omega)$ are both normal in $S(\Omega)$.

(c) Prove that neither of the groups $S(\Omega)$ and $S_{fin}(\Omega)$ is finitely generated. **Hint:** The two groups are not finitely generated for completely different reasons.

(d) Construct a finitely generated subgroup G of $S(\Omega)$ which contains $S_{fin}(\Omega)$. **Note:** This example shows that a subgroup of a finitely generated group does not have to be finitely generated.