Homework #3.

Approximate plan for next week: Direct and semi-direct products and further applications of Sylow theorems (5.4, 5.5).

Problems, to be submitted by Thursday, September 19th

1. Let G be a finite group, P a Sylow p-subgroup of G for some p and H a subgroup of G such that $N_G(P) \subseteq H \subseteq G$. Prove that $N_H(P) = N_G(P)$ and $[G : H] \equiv 1 \mod p$.

2. Prove that a group of order $132 = 3 \cdot 4 \cdot 11$ has a normal Sylow p-subgroup for some p.

3. Let G be a group of order 48. Prove that G has either a normal subgroup of order 16 or normal subgroups of orders 8 and 24.

4. Let G be a group of order 24. Prove that G has a normal 2-Sylow subgroup or a normal 3-Sylow subgroup or is isomorphic to S_4. **Note:** This is probably a pretty hard problem without a hint, but we will discuss how to approach it in class on Tue, Sep 17.

5. Let $\mathbb{F}_p$ be a finite field of order p, let $G = GL_n(\mathbb{F}_p)$ and $P = U_n(\mathbb{F}_p)$, the subgroup of upper-unitriangular matrices in $GL_n(\mathbb{F}_p)$.

(a) Prove that P is a Sylow p-subgroup of G (you may use the formula for $|G|$ established in Homework#2).

(b) Let $B = UT_n(\mathbb{F}_p)$, the subgroup of upper-triangular matrices in G. Prove that $N_G(P) = B$. **Hint:** To prove the inclusion $N_G(P) \subseteq B$ consider the natural action of G on $X = \mathbb{F}_p^n$. First compute $X^P = \{x \in X : gx = x \text{ for any } g \in P\}$, the set of fixed points of P. Now if $g \in N_G(P)$, what can you say about $g \cdot X^P$ (the image of X^P under the action of g). Use this result and induction on n to prove that $N_G(P) \subseteq B$.

(c) Find $n_p(G)$, the number of Sylow p-subgroups of G.

6. Explicitly describe a Sylow p-subgroup of S_{p^2}. See next page for a hint.

Hint for 6: The largest power of p dividing $(p^2)!$ is p^{p+1}. First find a subgroup of order p^p in S_{p^2} (this is quite easy) and then think how to enlarge it to a subgroup of order p^{p+1}.