Homework # 11.

Plan for the remaining classes: Dimension theory of affine varieties, localization of affine varieties and prime spectrum of a ring (parts of 15.2, 15.4 and 15.5). Good references on commutative algebra and algebraic geometry freely available online are notes by J. Milne

`http://www.jmilne.org/math/xnotes/CA.pdf`

and

`http://www.jmilne.org/math/CourseNotes/AG.pdf`

Problems, to be submitted by Thu, December 5th.

Problem 1: DF, Problem 19 on p. 332. Make sure to read about the Buchberger's algorithm in 9.6 prior to solving this problem.

Problem 2: Let I be an ideal of $\mathbb{Z}[x]$, and suppose that I contains a monic polynomial $f(x)$ of degree n. Prove that I can be generated (as an ideal) by at most $n+1$ elements.

Problem 3: Let k be a field. An algebraic set $V \subseteq k^n$ is called <u>irreducible</u> if $V \neq \emptyset$ and V cannot be written as the union $V = V_1 \cup V_2$ where V_1 and V_2 are both algebraic, with $V_1 \neq V$ and $V_2 \neq V$.

(a) (practice) Prove that V is irreducible if and only if its vanishing ideal $I(V)$ is prime.

(b) We will prove that any algebraic set V can be uniquely written as a union of finitely many algebraic subsets $V = \cup_{i=1}^{k} V_i$ where V_i's are irreducible and do not contain each other. Such V_i's are called <u>irreducible components of V</u>. Assume that k is infinite, and let

$$V = Z(xy - y, x^2z - z) \subset k^3,$$

the set of common zeroes of $xy - y$ and $x^2z - z$. Find irreducible components of V and their vanishing ideals. The answer will depend on $char(k)$.

Problem 4: Let k be an algebraically closed field. Prove that any nonzero prime ideal of $k[x, y]$ is equal to (f) for some irreducible $f \in k[x, y]$ or $(x - a, y - b)$ for some $a, b \in k$. You may use the fact that $k[x, y]$ has Krull dimension 2.

In Problems 5 and 6 we identify the set $Mat_n(k)$ of $n \times n$ matrices over a field k with k^{n^2} and thus can talk about Zariski topology on $Mat_n(k)$.

Problem 5: Let k be an algebraically closed field.

(a) Prove that $SL_n(k) = \{A \in Mat_n(k) : \det(A) = 1\}$ is Zariski closed (that is, closed in Zariski topology) and find its dimension.

(b) Fix $1 \leq d \leq n$, and let $R_d(n,k)$ be the set of all matrices in $Mat_n(k)$ which have rank $\leq d$. Prove that $R_d(n,k)$ is Zariski closed, guess its dimension and give a heuristic argument.

Problem 6: Let k be an arbitrary field. If Y is a subset of k^n, we will denote by $\overline{Y}$ the *Zariski closure* of Y, that is, the closure of Y in the Zariski topology.

Now let A be a commutative subset of $Mat_n(k)$, that is, $ab = ba$ for all $a, b \in A$. Prove that $\overline{A}$ is also commutative. **Hint:** First show that for any $a \in Mat_n(k)$, the centralizer of a in $Mat_n(k)$ is Zariski closed. Then show that $ab = ba$ for all $a \in A$ and $b \in \overline{A}$ and finally deduce the assertion of the problem.

Problem 7: Again let k be an algebraically closed field. Let Y be a subset of k^n. Let $k[Y] = k[x_1, \ldots, x_n]/I(Y)$. As we will discuss in class on Tue, Nov 26, $k[Y]$ can be naturally identified with the ring of polynomial functions from Y to k (with pointwise addition and multiplication). Let $O(Y)$ be the set of all everywhere defined rational functions on Y, that is, all functions $f : Y \to k$ for which there exist polynomials $p, q \in k[x_1, \ldots, x_n]$ s.t. q does not vanish at any point of Y and $f = p/q$ as a function on Y. Clearly, $k[Y] \subseteq O(Y)$.

(a) Prove that if Y is an algebraic set, then $O(Y) = k[Y]$. **Hint:** Use the weak Nullstellensatz.

(b) Let $Y = k^1 \setminus \{0\}$, the affine line with 0 removed. Prove that $k[Y] = k[x]$ (polynomials in one variable) while $O(Y) = k[x, 1/x]$.

(c) Find an algebraic subset Z of k^2 such that $k[Z] \cong k[x, 1/x]$. How is Z related to Y from part (b)? (No formal answer is expected).

(d) Find a non-algebraic subset W of k^2 for which $O(W) = k[W] \cong k[x_1, x_2]$.