Homework #9.

Plan for next week: Chapter 8 (Euclidean domains, principal ideal domains and unique factorization domains).

Problems, to be submitted by 1pm on Friday, November 9th

1. (a) Let G be a finitely generated group. Use Zorn's lemma to show that G has a maximal subgroup (recall that a maximal subgroup of G is a maximal element of the set of proper subgroups of G partially ordered by inclusion).
Hint: The key step is to show that if $\mathcal{C}$ is a chain of proper subgroups of G, then the union of subgroups in this chain is not the whole G.
(b) (optional) If your proof in (a) is correct, a nearly identical argument should imply that G always has a maximal normal subgroup. However, the latter is true under weaker assumptions on G. Can you find a natural condition on G (weaker than finite generation) that guarantees the existence of a maximal normal subgroup? Can you give an example of a group which has a maximal normal subgroup, but no maximal subgroup?

2. Let R be a commutative ring with 1. The *nilradical* of R denoted $Nil(R)$ is the set of all nilpotent elements of R, that is

$$Nil(R) = \{a \in R : a^n = 0 \text{ for some } n \in \mathbb{N}\}.$$

The *Jacobson radical* of R denoted by $J(R)$ is the intersection of all maximal ideals of R. Prove that
(a) $Nil(R)$ and $J(R)$ are ideals of R
(b) $Nil(R) \subseteq J(R)$.

Note: You will probably need to use the fact that every proper ideal I of R is contained inside some maximal ideal (in class we proved that any ring with 1 has a maximal ideal). This can be proved either by a simple modification of the proof from class or by applying the result from class to

3. (a) Problem 7.3.34. Note: in all exercises in 7.3 R is assumed to be a ring with 1 (this is crucial for this problem). Also note that IJ is NOT defined to be the set $\{ij : i \in I, j \in J\}$; by definition, IJ is the set of finite sums of elements of the form ij, with $i \in I, j \in J$.
(b) Read the section on the Chinese remainder theorem (7.6).

4. DF, Problem 7.1.26 and 7.1.27

5. (a) (practice) DF, Problem 7.2.3

(b) DF, Problem 7.2.5 (you may freely refer to 7.2.3).

(c) Let M be an ideal of a commutative ring R with 1. Prove that the following conditions are equivalent:

(i) M is the unique maximal ideal of R

(ii) every element of $R \setminus M$ is invertible.

A commutative ring with 1 which has the unique maximal ideal is called *local*.

6. Let $R = \mathbb{Z}_{14}$, $D = \{\bar{1}, \bar{2}, \bar{4}, \bar{8}\}$ (note that D is multiplicatively closed but it does contain zero divisors). Prove that the localization RD^{-1} is isomorphic to $\mathbb{Z}_7$.