Homework #8.

Plan for next week: We will start ring theory. I will first briefly go over the basics in 7.1-7.4 and then we will discuss rings of fractions and localizations (7.5).

Problems, to be submitted by Thursday, November 1st

1. Let G be the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by the matrices $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Find subsets A and B of $\mathbb{Z}^2$ satisfying the hypotheses of the Ping-Pong Lemma (and thereby prove that G is free of rank 2).

2. Let G be a group, S a subset of G and $H = \langle S \rangle$ the subgroup generated by S. One says that H is *freely generated by* S if the evaluation homomorphism $\varepsilon : F(S) \to H$ is an isomorphism.

Now let $G = F(a,b)$, the free group with 2 generators a and b, and let $S = \{aba^{-1}, a^2ba^{-2}, a^3ba^{-3}, \ldots\}$. Prove that $\langle S \rangle$ is free of countable rank (this gives another example of infinitely generated subgroup of a finitely generated group). Do not use the theorem that subgroups of free groups are free.

3. Let X be a set and $F(X)$ the free group on X. Given $f \in F(X)$, the *length of* f, denoted by $l(f)$ is defined to be the length of the unique reduced word in $X \cup X^{-1}$ representing f. Equivalently, $l(f)$ is the smallest n such that $f = x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$ for some $x_i \in X$ and $\varepsilon_i \in \{\pm 1\}$.

(a) Prove that for any $f \in F(X)$ there exist integers $a, b \in \mathbb{Z}_{\geq 0}$ (depending on f) such that $l(f^n) = na + 2b$ for any $n \in \mathbb{N}$. Describe explicitly (i.e. give an algorithm) how to compute a and b for a given f.

(b) Use (a) to show that free groups are torsion-free.

4.

(a) Explain why for any $n \in \mathbb{N}$ there are only finitely many isomorphism classes of groups of order n.

(b) Let G be a finitely generated group and H a finite group. Prove that there are only finitely many homomorphisms from G to H. **Hint:**

A homomorphism from G is completely determined by its values on generators.

(c) Let G be a finitely generated group. Prove that for any $n \in \mathbb{N}$ there are only finitely many normal subgroups of index n in G. Then deduce that G has only finitely many subgroups of index n (use the small index lemma).

5. Let p and q be primes with $p < q$ and $q \equiv 1 \mod p$, and let G be a non-abelian group or order pq. Recall that such G is unique up to isomorphism. Prove that G has a presentation $\langle x, y \mid x^p = 1, y^q = 1, xyx^{-1} = y^a \rangle$ where a is coprime to q and the order of $\bar{a}$ in $\mathbb{Z}_q$ is equal to p.

Hint: Let $\widehat{G} = \langle x, y \mid x^p = 1, y^q = 1, xyx^{-1} = y^a \rangle$. By definition $\widehat{G}$ is the quotient of $F(x, y)$ by the normal closure of the set $\{x^p, y^q, xyx^{-1}y^{-a}\}$. To prove the statement first show that G has elements X and Y satisfying the above 3 relations; then show that there is a surjective homomorphism $\phi : \widehat{G} \to G$ such that $\phi(x) = X$ and $\phi(y) = Y$ (you will need the universal property of free groups for this). Finally, prove that $|\widehat{G}| \leq pq$ and deduce that ϕ is an isomorphism.